Existence of Einstein metrics on Fano manifolds

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This is largely an expository paper and dedicated to my friend J. Cheeger for his 65th birthday. The purpose of this paper is to discuss some of my works on the existence of Kähler-Einstein metrics on Fano manifolds and some related topics. I will describe a program I have been following for the last twenty years. It includes some of my results and speculations which were scattered in my previous publications or mentioned in my lectures. I also take this opportunity to clarify and make them more accessible. In the course of doing so, I will also discuss some recent advances and problems which arise from studying the existence problem.

A Fano manifold is a compact Kähler manifold with positive first Chern class. It has been one of main problems in Kähler geometry to study if a Fano manifold admits a Kähler-Einstein metrics since the Aubin-Yau theorem on Kähler-Einstein metrics with negative scalar curvature and the Calabi-Yau theorem on Ricci-flat Kähler metrics in 70's. This problem is much more difficult because there are new obstructions to the existence. The classical one was given by Matsushima: If a Fano manifold M admits a Kähler-Einstein metric, then its Lie algebra of holomorphic vector fields must be reductive. In the early 80's, A. Futaki introduced a new invariant, now referred to the Futaki invariant, whose vanishing is a necessary condition for M to have a Kähler-Einstein metric. Since late 80's, inspired by my works on Kähler-Einstein metrics on complex surfaces [Ti89], I have been developing methods of relating certain geometric stability of underlying manifolds to Kähler-Einstein metrics. In [Ti97], I introduced the K-stability for any Fano manifold and proved that a Fano manifold with trivial holomorphic vector fields and which admits a Kähler-Einstein metric is K-stable. An algebraic version of the K-stability was given by Donaldson in [Do02]. It was conjectured that the existence of Kähler-Einstein metrics on M is equivalent to the asymptotic K-stability.[1]

As said at the beginning, this is not intended to be a complete survey on Kähler-Einstein metrics with positive scalar curvature. Unfortunately, there are

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[1]The K-stability can be extended to any polarized projective algebraic manifolds and similar results can be proved for general Kähler metrics with constant scalar curvature. Also a conjecture, often referred as the Yau-Tian-Donaldson conjecture, can be made on existence of Kähler metrics with constant scalar curvature and the K-stability. I will refer the readers to [Ti97], [Do00], [Sto07] for more discussions and also to Section 4.1 (Conjecture 4.7) below.

important works which I can not present here because of limited time and space. However, I do plan to write a much more complete survey and hope to include many of them there.

Contents

1 Preliminary

Let M be a Fano manifold of dimension n. A Kähler metric can be given by specifying its Kähler form ω, in local coordinates $z_1, \cdots, z_n$, it is of the form

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $\{g_{i\bar{j}}\}$ is a positive Hermitian matrix-valued function such that $d\omega = 0$. We will simply use ω to denote both a metric and its Kähler form.

Recall that the Kähler class of ω is the cohomology class $[\omega]$ in $H^2(M, \mathbb{R})$ represented by ω. It follows from Hodge theory that if ω' is another Kähler metric with $[\omega'] = [\omega]$, then there is a smooth function φ on M such that

$$\omega' = \omega + \frac{\sqrt{-1}}{2} \partial\bar{\partial}\varphi.$$

We will often denote the right side by ω_φ. Thus, the space $\mathcal{K}_{[\omega]}$ of Kähler metrics with the same Kähler class $[\omega]$ can be identified with

$$\{\varphi \in C^\infty(M,\mathbb{R}) \mid \int_M \varphi\, \omega^n = 0,\ \omega + \frac{\sqrt{-1}}{2}\partial\bar\partial\varphi > 0\}.$$

Since M is Fano, we can take ω such that $[\omega] = \pi\, c_1(M)$. We will assume this unless specified. In this paper, we call ω a Kähler-Einstein metric if $\mathrm{Ric}(\omega) = \omega$, where $\mathrm{Ric}(\omega)$ is the Ricci curvature form of ω, in local coordinates $z_1, \cdots, z_n$,

$$\mathrm{Ric}(\omega) = \frac{\sqrt{-1}}{2}\partial\bar\partial \log\det(g_{i\bar j}).$$

The following uniqueness theorem is due to Bando and Mabuchi [BM86].

Theorem 1.1. *Any given compact Fano manifold M admits at most one Kähler-Einstein metric up to automorphisms.*

The main concern of this paper is on the existence. First we recall an analytic obstruction introduced by Futaki in 1983. Let $\eta(M)$ be the space of holomorphic vector fields on M and ω be any fixed Kähler metric with $\pi\, c_1(M)$ as its Kähler class $\Omega \in H^2(M,\mathbb{R})$. We put

$$f_\Omega(v) = \int_M v(h_\omega)\omega^n, \quad v \in \eta(M),$$

where h_ω is determined by the equations

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2}\partial\bar\partial h_\omega, \quad \int_M \left(e^{h_\omega} - 1\right)\,\omega^n = 0,$$

where $\underline{s}_\omega$ denotes the average of $s(\omega)$. It was proved in [Fut83] that $f_\Omega(v)$ is actually independent of the choice of ω. Therefore, it is an invariant, referred as the Futaki invarant. Consequently, if M admits a Kähler-Einstein metric, then $f_\Omega \equiv 0$. On the other hand, there are examples of Fano manifolds with non-vanishing Futaki invariant, so there do not exist Kähler-Einstein metrics on such manifolds.

As usual, one can reduce the existence of Kähler-Einstein metrics to solving a complex Monge-Ampere equation. Let ω be a Kähler metric and h_ω be defined as above. Then ω_φ is a Kähler-Einstein metric if and only if modulo a constant, φ solves the following

$$(\omega + \frac{\sqrt{-1}}{2}\partial\bar\partial\varphi)^n \ = \ e^{h_\omega - \varphi}\omega^n. \tag{1.1}$$

There are two ways of solving this equation. One is to use the Kähler-Ricci flow. We will discuss this method in details in later sections. Another one is the continuity method. We will first discuss this continuity method. Consider

$$(\omega + \frac{\sqrt{-1}}{2}\partial\bar\partial\varphi)^n \ = \ e^{h_\omega - t\varphi}\omega^n. \tag{1.2}$$

Set $I = \{t \in [0,1] \mid (1.2)$ is solvable for $s \in [0,t]\ |$. Using Yau's solution for the Calabi conjecture [Ya76], there is a solution φ for (1.2) with $t = 0$, so $0 \in I$, i.e., I is non-empty. It was observed in [Au83] that I is open. Its proof can be outlined as follows: If φ is a solution of (1.2) for some $t_0 < 1$, then a direct computation shows that $\mathrm{Ric}(\omega_\varphi) > t_0\omega_\varphi$. So it follows from the Bochner identity that the Laplacian Δ_{t_0} of ω_φ has non-zero eigenvalue greater than t_0. On the other hand, the linearization of (1.2) is simply $\Delta_{t_0} + t_0$ has non-zero eigenvalue, so it is invertible. Then the openness follows from the Implicit Function Theorem. To prove that I is closed, we need an a prior $C^{2,\alpha}$-estimate for any solutions of (1.2). In view of the Aubin-Yau's 2nd order estimates (cf. [Au76], [Ya76]) and Calabi's 3rd estimate (cf. Appendix in [Ya76]), we only need to derive an a prior C^0-estimate for any solutions of (1.2).

Since there are analytic obstructions, such a C^0-estimate relies on geometry of underlying M.

2 Analytic Criterion for Existence

We have seen that in order to prove the existence of Kähler-Einstein metrics on M, we need some extra condition for M to get an a priori C^0-estimate. One analytic condition can be formulated in terms of the properness of the Lagrangian of (1.1).

The space of Kähler metrics with a fixed Kähler class $[\omega] = \pi\, c_1(M)$ is

$$\mathcal{K}_{[\omega]} = P(M,\omega)/\sim, \quad P(M,\omega) = \{\varphi \in C^\infty(M,\mathbb{R}) \mid \omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi > 0\}.$$

Here $\varphi \sim \varphi'$ means $\varphi = \varphi' + c$ for some constant c.

Define

$$\mathbf{I}_\omega(\varphi) = \frac{1}{V}\int_M \varphi(\omega^n - (\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi)^n), \tag{2.1}$$

where $V = \int_M \omega^n$. Also define

$$\mathbf{J}_\omega(\varphi) = \int_0^1 \frac{\mathbf{I}_\omega(t\varphi)}{t}dt = \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\frac{\sqrt{-1}}{2}\int_M \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i \wedge \omega_\varphi^{n-i-1}. \tag{2.2}$$

Then

$$\mathbf{I}_\omega(\varphi) - \mathbf{J}_\omega(\varphi) = \frac{1}{V}\sum_{i=0}^{n-1}\frac{n-i}{n+1}\frac{\sqrt{-1}}{2}\int_M \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i \wedge \omega_\varphi^{n-i-1}.$$

Notice that for any $\varphi \in P(M,\omega)$,

$$\mathbf{I}_\omega(\varphi) \ \geq\ 0, \mathbf{J}_\omega(\varphi)\ \geq\ 0, \mathbf{I}_\omega(\varphi) - \mathbf{J}_\omega(\varphi)\ \geq\ 0.$$

Moreover, one can deduce from the definition

Lemma 2.1. *For any $\varphi \in P(M,\omega)$, we have*

$$\frac{1}{n}\,\mathbf{J}_\omega(\varphi) \;\le\; \mathbf{I}_\omega(\varphi) - \mathbf{J}_\omega(\varphi) \;\le\; n\,\mathbf{J}_\omega(\varphi).$$

The Lagrangian of (1.1) is given by

$$\mathbf{F}_\omega(\varphi) = \mathbf{J}_\omega(\varphi) - \frac{1}{V}\int_M \varphi\omega^n - \log\left(\frac{1}{V}\int_M e^{h_\omega - \varphi}\omega^n\right). \tag{2.3}$$

In this section, we give a sufficient condition for the existence of Kähler-Einstein metrics on a Fano manifold in terms of the properness of $\mathbf{F}$. We will also show that such a condition is necessary for Fano manifolds without non-trivial holomorphic fields.

2.1 Analytic stability

In this subsection, we introduce a notion of analytic stability and discuss its simple implications.

Definition 2.2. *We say that a Fano manifold M is analytically stable if $\mathbf{F}_\omega$ is proper, that is, if there is an increasing function $f(t) \ge -c$ for some $c \ge 0$, where $t \in (-\infty,\infty)$, such that $\lim_{t\to\infty} f(t) = \infty$ and for any $\varphi \in P(M,\omega)$, we have*[2]

$$\mathbf{F}_\omega(\varphi) \ge f(\mathbf{I}_\omega(\varphi) - \mathbf{J}_\omega(\varphi)). \tag{2.4}$$

We say that M is analytically semi-stable if $\mathbf{F}_\omega$ is bounded from below.

If $\omega' = \omega_\psi$ is another Kähler metric, then one can show

$$\mathbf{F}_\omega(\varphi) = \mathbf{F}_{\omega'}(\varphi - \psi) + \mathbf{F}_\omega(\psi). \tag{2.5}$$

Hence, the analytic stability is independent of the choice of the base metric ω. Clearly, the analytic stability also implies the semi-stability.

Denote by $\mathrm{Aut}(M)$ the group of all holomorphic automorphisms of M and by $\eta(M)$ its Lie algebra.

Remark 2.3. *If G is a compact Lie group acting on M by automorphisms of $(M,[\omega])$ and ω is a G-invariant metric in $[\omega]$, then we say that $(M,[\omega])$ is analytically G-stable if (2.8) holds for all G-invariant φ in $P(M,\omega)$. Similarly, one can define analytic G-semi-stability.*

For any $\sigma \in \mathrm{Aut}(M)$ and $\varphi \in P(M,\omega)$, there is a unique φ_σ such that

$$\sigma^*\omega_\varphi = \omega + \frac{\sqrt{-1}}{2}\partial\bar\partial\varphi_\sigma \quad \text{and} \quad \int_M \left(e^{h_\omega - \varphi_\sigma} - 1\right)\omega^n \;=\; 0.$$

Let $X \in \eta(M)$ be a holomorphic vector field, then there is a unique θ_X satisfying:

$$i_X\omega = \frac{\sqrt{-1}}{2}\bar\partial\theta_X \quad \text{and} \quad \int_M \theta_X\, e^{h_\omega}\,\omega^n = 0.$$

[2]By Lemma 2.1, one can replace $\mathbf{I} - \mathbf{J}$ by $\mathbf{J}$ in the definition.

Since X is holomorphic, $\nabla^{0,1}\bar{\partial}\theta_X = 0$. It follows from this and the above that

$$\Delta_\omega \theta_X + \theta_X + X(h_\omega) \,=\, 0. \tag{2.6}$$

If $\sigma(t)$ is a one-parameter subgroup generated by the real part $\mathrm{Re}(X)$ of X, then $\dot{\varphi}_{\sigma(t)} = \theta_X(t,\cdot)$, where $\theta_X(t)$ is the corresponding potential θ_X of X when ω is replaced by ω_φ for $\varphi = \varphi_{\sigma(t)}$. It follows

$$\frac{d\mathbf{F}_\omega}{dt}(\varphi_{\sigma(t)}) = \frac{1}{V}\int_M \dot{\varphi}_{\sigma(t)}\,\sigma^*\omega_\varphi^n \,=\, -\,\mathrm{Re}\left(\frac{1}{V}\int_M X(h_{\sigma^*\omega_\varphi})\,\sigma^*\omega_\varphi^n\right). \tag{2.7}$$

The last integral is simply the Futaki invariant and is independent of ω_φ [Fut83]. Therefore, we have

Corollary 2.4. *If M is analytic semi-stable, then the Futaki invariant vanishes.*

Let $\mathrm{Aut}_0(M)$ be the connected component of $\mathrm{Aut}(M)$ containing the identity. If $\mathrm{Aut}_0(M)$ is non-trivial, then by the above corollary, M can not be analytic stable. Hence, we need the following extension of the analytic stability.

Definition 2.5. *We say that a Fano manifold M is weakly analytically stable if $\mathrm{Aut}_0(M)$ is reductive*[3] *and there is an increasing function $f(t) \geq -c$ for some $c \geq 0$, where $t \in (-\infty,\infty)$, such that $\lim_{t\to\infty} f(t) = \infty$ and for any $\varphi \in P(M,\omega)$, we have*

$$\mathbf{F}_\omega(\varphi) \,\geq\, \inf_{\sigma\in \mathrm{Aut}_0(M)} f(\mathbf{I}_\omega(\varphi_\sigma) - \mathbf{J}_\omega(\varphi_\sigma)). \tag{2.8}$$

Of course, weak analytic stability implies the semi-stability and is independent of the choice of the base Kähler metric ω.

The above notion of analytic stability can be also defined by the K-energy in place of $\mathbf{F}$. The K-energy was introduced by T. Mabuchi in [Ma86]. It is defined as follows: For any $\varphi \in P(M,\omega)$, let φ_t be any path joining 0 to φ, then

$$\mathbf{T}_\omega(\varphi) = -\frac{1}{V}\int_0^1\int_M \dot{\varphi}_t(s(\omega_{\varphi_t}) - \underline{s}_\omega)\,\omega_{\varphi_t}^n\,dt, \tag{2.9}$$

where $s(\cdot)$ denotes the scalar curvature and $\underline{s}$ denotes the average of scalar curvature. It is not hard to show that $\mathbf{T}_\omega(\varphi)$ is independent of the choice of the path. Also, as $\mathbf{F}$, we have the following cocycle condition for $\mathbf{T}$:

$$\mathbf{T}_\omega(\varphi) + \mathbf{T}_{\omega_\varphi}(\psi) = \mathbf{T}_\omega(\varphi+\psi).$$

The analytic stability using $\mathbf{T}$ is equivalent to the older one using $\mathbf{F}$. One direction follows easily from the following identiy proved in [DT91]:

$$\mathbf{F}_\omega(\varphi) \,=\, \mathbf{T}_\omega(\varphi) + \frac{1}{V}\int_M h_{\omega_\varphi}\omega_\varphi^n - \frac{1}{V}\int_M h_\omega \omega^n.$$

[3] This may follow from the second condition imposed on $\mathbf{F}_\omega$

Since $\int_M e^{h_{\omega_\varphi}}\omega_\varphi^n = V$, we have $\int_M h_{\omega_\varphi}\omega_\varphi^n \le 0$ by the concavity of logarithm. The other direction is more tricky and follows from the results of [Ti97] by proving that the analytic stability defined by either F or T is equivalent to existence. The fact that the analytic semi-stability defined by F is equivalent to that defined by T was proved by Y. Rubinstein[Ru07].

One advantage of using **T** is that one can extend the notion of analytic stability to any polarized Kähler manifolds which are not necessarily Fano. A polarized Kähler manifold (M,Ω) is a compact Kähler manifold together with a Kähler class Ω. It was conjectured in [Ti98] that *a compact Kähler manifold M admits a Kähler metric of constant scalar curvature in the Kähler class Ω if the polarized manifold (M,Ω) is analytically stable. The converse is also true if* $\mathrm{Aut}_0(M,\Omega)$ *is trivial or in general cases, it holds in a suitable notion of analytic stability.* This conjecture was solved in the case of Fano manifolds [Ti97] as we will discuss in the following subsections.

2.2 From properness to existence

In this subsection, we prove the following

Theorem 2.6. *Let M be a weakly analytically stable Fano manifold. Then M admits a Kähler-Einstein metric.*

This theorem is a slight generalization of a known result (cf. [Ti97]): Any Fano manifold which is analytically stable admits a Kähler-Einstein metric. A converse to this result is proved in [Ti97]. In a later subsection, we will prove a converse to the above theorem.

The rest of this subsection is devoted to giving an outlined proof of Theorem 2.6. We will skip the arguments in the proof which have now become standard. We refer the readers to [Ti98] for those arguments. First we notice that we may assume ω is K-invariant for a maximal compact subgroup K of $\mathrm{Aut}_0(M)$.

As usual, we apply the continuity method to (1.2). By Yau's solution to the Calabi conjecture and the Implicit Function Theorem, we have a solution φ_t to (1.2) for each sufficiently small $t > 0$. By an observation of T. Aubin (cf. [Au83], also [Ti98]), the set S of t for which (1.2) admits a solution is open in [0,1]. So it suffices to show that S is closed. This amounts to establishing an a prior C^3-estimate for solutions φ_t of (1.2). On the other hand, by Aubin-Yau's C^2-estimate and Calabi's C^3-estimate, we only need to prove an a priori C^0-estimate. For this, we need to use the weak analytic stability.

Let φ_t be a solution of (1.2):

$$(\omega + \frac{\sqrt{-1}}{2}\partial\bar\partial\varphi_t)^n \;=\; e^{h_\omega - t\varphi_t}\omega^n.$$

We may assume that $t < 1$ and write $\omega_t = \omega + \frac{\sqrt{-1}}{2}\partial\bar\partial\varphi_t$. Taking $\partial\bar\partial$-derivative on both sides of the above, we obtain a known bound:

$$\mathrm{Ric}(\omega_t) \;=\; t\omega_t + (1-t)\omega \;>\; t\omega_t. \tag{2.10}$$

Hence, Ricci curvature of ω_t is bounded from below by t. It follows that there is a uniform Sobolev inequality and the Poincare inequality for ω_t. Then by using the Moser iteration (cf. [Ti87]), we get[4]

Lemma 2.7. *There is a uniform constant $c > 0$ such that for any solution φ_t of* (1.2) *($t \in [0,1]$),*

$$||\varphi_t||_{C^0} \ \le\ c(1 + \mathbf{J}_\omega(\varphi_t)).$$

So we only need to bound $\mathbf{J}_\omega(\varphi_t)$ or equivalently, $\mathbf{I}_\omega(\varphi_t) - \mathbf{J}_\omega(\varphi_t)$. It was shown in [Ti87]: If $\phi(s)$ be a smooth variation of $\phi \in P(M,\omega)$, i.e., $\phi(0) = \phi$, then

$$\frac{d}{ds}\left(\mathbf{I}_\omega(\phi(s)) - \mathbf{J}_\omega(\phi(s))\right)\big|_{s=0} \ =\ -\frac{1}{V}\int_M \phi\,\Delta\dot{\phi}\,\omega_\phi^n, \tag{2.11}$$

where $\dot{\phi} = \frac{\partial\phi}{\partial s}$ at $s = 0$. Then we have

Lemma 2.8. *Let $X \in \eta(M)$ and $\sigma(s)$ be an one-parameter subgroup in* $\mathrm{Aut}_0(M)$ *generated by the real part of X and such that $\sigma(0) = \mathrm{Id}$, then*

$$\frac{d}{ds}\left(\mathbf{I}_\omega(\phi_{\sigma(s)}) - \mathbf{J}_\omega(\phi_{\sigma(s)})\right)\big|_{s=0} \ =\ \mathrm{Re}\left(\frac{1}{V}\int_M X(\phi)\,\omega_\phi^n\right). \tag{2.12}$$

Proof. Differentiating $\sigma(s)^*\omega_\phi = \omega_{\phi(s)}$ on s, where $\phi(s) = \phi_{\sigma(s)}$, we obtain

$$d i_X\omega_\phi \ =\ \frac{\sqrt{-1}}{2}\partial\bar\partial\dot{\phi}.$$

On the other hand, as above, there is a unique θ satisfying:

$$i_X\omega_\phi = \frac{\sqrt{-1}}{2}\bar\partial\theta \quad \text{and} \quad \int_M \theta\, e^{h_{\omega_\phi}}\,\omega_\phi^n \ =\ 0.$$

Then we have $\dot{\phi} = \mathrm{Re}(\theta + C)$. Plugging this into (2.11) and integrating by parts, we easily deduce (2.12). □

Lemma 2.9. *For any $\phi \in P(M,\omega)$, the function $\mathbf{I}_\omega(\phi_\sigma) - \mathbf{J}_\omega(\phi_\sigma)$ defined on the* $\mathrm{Aut}_0(M)$ *is K-invariant and convex.*

Proof. The K-invariance follows from

$$\mathbf{I}_\omega(\phi_\sigma) - \mathbf{J}_\omega(\phi_\sigma) = \mathbf{I}_{\sigma^*\omega}(\phi_\sigma) - \mathbf{J}_{\sigma*\omega}(\phi_\sigma) = \mathbf{I}_\omega(\phi) - \mathbf{J}_\omega(\phi),$$

where $\sigma \in K$. For the convexity, we first observe that being reductive, $\eta(M)$ is generated by holomorphic vector fields X satisfying: $i_X\omega = \frac{\sqrt{-1}}{2}\bar\partial\theta$ for some real-valued function θ. The condition means that $\mathrm{Im}(X)$ is a Killing field. Let $\sigma(s)$ be the one-parameter subgroup generated by $\mathrm{Re}(X)$. Write $\phi(s,x) = \phi_{\sigma(s)}(x)$, then one can easily show

$$\frac{\partial\phi}{\partial s}(s,x) \ =\ \frac{\partial\phi}{\partial s}(0,\sigma(s)(x)).$$

[4] One can also use the arguments as those in [BM86] by using the Green function.

It follows that

$$\frac{\partial^2 \phi}{\partial s^2} = d\left(\frac{\partial \phi}{\partial s}\right)(\mathrm{Re}(X)) = \left|\partial\left(\frac{\partial \phi}{\partial s}\right)\right|^2_{\omega_\phi}.$$

Here we need to use the fact that θ is real. Hence, $\phi_{\sigma(s)}$ is a geodesic with respect to the L^2-metric on the space of Kähler metrics. Then the convexity follows from an observation of X.X. Chen. For the readers' convenience, we sketch a proof as follows: As shown in [Ti87] (also see (2.12)), we have

$$\frac{d}{ds}\left(\mathbf{I}_\omega(\phi_{\sigma(s)}) - \mathbf{J}_\omega(\phi_{\sigma(s)})\right) = \frac{1}{V}\int_M \dot\phi\,(\omega - \omega_\phi) \wedge \omega_\phi^{n-1},$$

where $\dot\phi = \frac{\partial \phi}{\partial s}$. This implies

$$\begin{aligned} &\frac{d^2}{ds^2}\left(\mathbf{I}_\omega(\phi_{\sigma(s)}) - \mathbf{J}_\omega(\phi_{\sigma(s)})\right) \\ = \; &\frac{1}{V}\int_M |\partial\dot\phi|^2_{\omega_{\phi(s,\cdot)}}\,\omega\wedge\omega^{n-1}_{\phi(s,\cdot)} - (n-1)\frac{\sqrt{-1}}{2}\,\partial\dot\phi\wedge\bar\partial\dot\phi\wedge\omega\wedge\omega^{n-2}_{\phi(s,\cdot)}. \end{aligned}$$

It is easy to see that the integrand in the above integral is non-negative. The lemma is proved. □

Corollary 2.10. *For any solution $\phi = \varphi_t$ of (1.2) with $t < 1$, the minimum of $\mathbf{I}_\omega(\phi_\sigma) - \mathbf{J}_\omega(\phi_\sigma)$ is attained at $\sigma = \mathrm{Id}$.*

Proof. It follows from (2.10)

$$h_{\omega_t} = -(1-t)\varphi_t + a_t. \tag{2.13}$$

Combining this with (2.12), we get

$$\frac{d}{ds}\left(\mathbf{I}_\omega(\phi_{\sigma(s)}) - \mathbf{J}_\omega(\phi_{\sigma(s)})\right)\Big|_{s=0} = -\mathrm{Re}\left(\frac{1}{(1-t)V}\int_M X(h_{\omega_t})\,\omega_\phi^n\right). \tag{2.14}$$

The integral on the right is simply the Futaki invariant. Since M is analytically semi-stable, the Futaki invariant vanishes identically. Therefore, φ_t is a critical point of $\mathbf{I}_\omega - \mathbf{J}_\omega$ restricted to the orbit of φ_t by $\mathrm{Aut}_0(M)$, so by last lemma, it has to be the minimum. □

It follows from the above corollary and the definition of weak analytic stability, we obtain

$$\mathbf{F}_\omega(\varphi_t) \geq f(\mathbf{I}_\omega(\varphi_t) - \mathbf{J}_\omega(\varphi_t)).$$

But it is known that $\mathbf{F}_\omega$ is bounded from above, so $\mathbf{I}_\omega(\varphi_t) - \mathbf{J}_\omega(\varphi_t)$ is bounded. The theorem is proved.

2.3 From existence to properness

In this subsection, we give a converse to Theorem 2.6. This is amount to proving a differential inequality. For simplicity, we discuss only the case that $\eta(M) = \{0\}$. We will leave the general case to a future paper since the proof is more involved.

Theorem 2.11. ([Ti97]) *Let (M, ω) be a Kähler-Einstein manifold with $\mathrm{Ric}(\omega) = \omega$ and without any non-trivial holomorphic vector fields. Then there are $c > 0$ and $\epsilon \in (0, 1)$*[5] *such that for any $\varphi \in P(M, \omega)$, we have*

$$\mathbf{F}_\omega(\varphi) \geq \frac{\mathbf{J}_\omega(\varphi)}{c(1 + ||\varphi||_{C^0})^{1-\epsilon}} - c. \tag{2.15}$$

This theorem was first given in [Ti97]. In the following, we will outline its proof. The readers can find all the details in [Ti97].

Let φ be given in the theorem and write $\omega' = \omega_\varphi$. The following lemma is well-known and was first proved by Bando-Mabuchi ([BM86], also see [Ti97]).

Lemma 2.12. *There is a unique family $\{\varphi_t\} \subset P(M, \omega')$ such that $\varphi_1 = -\varphi$ and*

$$(\omega' + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi_t)^n = e^{h_{\omega'} - t\varphi_t}\, \omega'^n.$$

Now we rewrite the equation in the above lemma as

$$(\omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\psi_t)^n = e^{(1-t)\varphi_t - \psi_t}\omega^n,$$

where $\psi_t = \varphi_t - \varphi_1 = \varphi_t + \varphi$.

Put $\omega_t = \omega' + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi_t = \omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\psi_t$, then $\omega_1 = \omega$ and

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega' \geq t\omega_t, \quad h_{\omega_t} = -(1-t)\varphi_t + a_t,$$

where a_t is chosen such that

$$\int_M \left(e^{-(1-t)\varphi + a_t} - 1\right)\omega_t^n = 0.$$

It follows that $|a_t| \leq (1-t)||\varphi_t||_{C^0}$ and $\Delta_{\omega_t} h_{\omega_t} + n\,(1-t) > 0$.

Applying Proposition 3.1 in [Ti97] to each ω_t, we obtain a new Kähler metric $\omega'_t = \omega_t + \partial\bar{\partial}u_t$ satisfying:

$$||u_t||_{C^0} \leq 3(1-t)||\varphi_t||_{C^0}, \tag{2.16}$$

$$||h_{\omega'_t}||_{C^{\frac{1}{2}}} \leq C(n, \omega'_t)(1 + (1-t)^2||\varphi_t||^2_{C^0})^{n+1}(1-t)^\beta, \tag{2.17}$$

where $\beta = \beta(n) \in (0, 1)$ and $C(n, \omega'_t)$ depends only on the first non-zero eigenvalue and the Sobolev constant of (M, ω'_t). Note that ω'_t is obtained by evolving ω_t along the Kähler-Ricci flow in time 1 (see [Ti97], section 3).

[5] This ϵ depends only on n. It is possible to give an explicit estimate on ϵ by examining the arguments in the proof. However, this is inessential.

Choose μ_t such that

$$\int_M \left(e^{h_{\omega_t}-u_t+\mu_t}-1\right)\omega_t^n = 0.$$

Then $|\mu_t| \le 6(1-t)||\varphi_t||_{C^0}$.

An application of the maximum principle implies

$$\varphi_t = \varphi_1 - w_t - u_t + \mu_t + a_t, \tag{2.18}$$

where w_t is the unique solution of the following $\omega = \omega_t' + \partial\overline{\partial} w_t$ and

$$\omega^n = e^{h_{\omega_t'} - w_t}\omega_t'^n. \tag{2.19}$$

Hence, φ_t is uniformly equivalent to φ_1 as long as w_t is uniformly bounded.

Define

$$\Phi_t(w) = \log\left(\frac{(\omega_1 - \partial\overline{\partial} w)^n}{\omega_1^n}\right) + h_{\omega_t'} - w. \tag{2.20}$$

Clearly, $\Phi_t : C^{2,\frac{1}{2}}(M) \mapsto C^{0,\frac{1}{2}}(M)$. A direct computation shows that the linearization of Φ at $w = 0$ is $-\Delta - 1$. Since $\eta(M) = \{0\}$, using the standard Bochner identity, one can show that it is invertible. Then by the Inverse Function Theorem, there is a $\delta > 0$ such that $\Phi_t(w) = 0$ has a unique solution w whenever

$$||h_{\omega_t'}||_{C^{\frac{1}{2}}(\omega_1)} < \delta,$$

furthermore, we have

$$||w||_{C^{2,\frac{1}{2}}(\omega_1)} \le C\delta.$$

Here C denotes a uniform constant.

Note that $C(n, \tilde{\omega})$ is uniformly bounded by a constant $\bar{c}$ for any metric $\tilde{\omega}$ satisfying:

$$\frac{1}{2}\omega \le \tilde{\omega} \le 2\omega.$$

.

Now we choose t_0 such that

$$\begin{aligned}
\delta' &= (1-t_0)^\beta(1+(1-t_0)^2||\varphi_{t_0}||^2_{C^0})^{n+1} \\
&= \sup_{t_0 \le t \le 1}(1-t)^\beta(1+(1-t)^2||\varphi_t||^2_{C^0})^{n+1},
\end{aligned} \tag{2.21}$$

where $\bar{c}\delta' = \delta$. We may further assume that $C\delta < \frac{1}{4}$. Then for any $t \in [t_0, 1]$,

$$||w_t||_{C^{2,\frac{1}{2}}(\omega_1)} < \frac{1}{4}.$$

Summarizing the above estimates, we can deduce that for any $t \in [t_0, 1]$,

$$||\varphi_t||_{C^0} \ge (1 - 10(1-t))||\varphi_1||_{C^0} - 1. \tag{2.22}$$

Now we recall an identity due to Ding-Tian:

$$t\left(\mathbf{J}_{\omega'}(\varphi_t) - \frac{1}{V}\int_M \varphi_t \omega'^n\right) \;=\; -\int_0^t \left(\mathbf{I}_{\omega'}(\varphi_s) - \mathbf{J}_{\omega'}(\varphi_s)\right) ds. \tag{2.23}$$

It follows

$$\mathbf{F}_{\omega'}(\varphi_1) \;=\; -\int_0^1 \left(\mathbf{I}_{\omega'}(\varphi_t) - \mathbf{J}_{\omega'}(\varphi_t)\right) dt.$$

Since $I_\omega(\varphi_t) - J_\omega(\varphi_t)$ is nondecreasing, we deduce from this

$$\begin{aligned} \mathbf{F}_\omega(\varphi) &= -\mathbf{F}_{\omega'}(\varphi_1) \qquad\qquad (2.24)\\ &\geq \min\{1-t_0, \frac{1}{12}\}\ (\mathbf{I}_{\omega'}(\varphi_{t_0}) - \mathbf{J}_{\omega'}(\varphi_{t_0}))\\ &\geq \min\{1-t_0, \frac{1}{12}\}\ (\mathbf{I}_{\omega'}(\varphi_1) - \mathbf{J}_{\omega'}(\varphi_1)) - 20(1-t_0)^2||\varphi_{t_0}||_{C^0} - 2. \end{aligned}$$

Then the theorem follows from this last inequality by some simple manipulation.

Since $J_\omega(\varphi)$ and $||\varphi||_{C^0}$ bound each other along (1.2), Theorem 2.11 can be regarded as a converse to Theorem 2.6. However, we can do better and get a genuine converse. In the following, we present a result of Tian-Zhu from [TZ97], which improves Theorem 2.11.

By (2.23) and the monotonicity of $\mathbf{I}_{\omega'} - \mathbf{J}_{\omega'}$, we get

$$\mathbf{F}_{\omega'}(\varphi_1) - t\left(\mathbf{J}_{\omega'}(\varphi_t) - \frac{1}{V}\int_M \varphi_t\omega'^n\right) \;\geq\; -(1-t)\left(\mathbf{I}_{\omega'}(\varphi_1) - \mathbf{J}_{\omega'}(\varphi_1)\right). \tag{2.25}$$

By the concavity of the logarithm, we have

$$-\log\left(\frac{1}{V}\int_M e^{h_{\omega'}-\varphi_t}\omega'^n\right) \;\leq\; \frac{1-t}{V}\int_M \varphi_t\omega_t^n.$$

Putting this and (2.25) together, we get

$$\begin{aligned} \mathbf{F}_\omega(\varphi_t - \varphi_1) &= \mathbf{F}_{\omega'}(\varphi_t) - \mathbf{F}_{\omega'}(\varphi_1) \qquad\qquad (2.26)\\ &\leq (1-t)\left(\mathbf{I}_{\omega'}(\varphi_1) - \mathbf{J}_{\omega'}(\varphi_1) - \mathbf{I}_{\omega'}(\varphi_t) + \mathbf{J}_{\omega'}(\varphi_t)\right)\\ &\leq (1-t)(\mathbf{I}_{\omega'}(\varphi_1) - \mathbf{J}_{\omega'}(\varphi_1)) \;=\; (1-t)\,\mathbf{J}_\omega(\varphi). \end{aligned}$$

Next we recall the following lemma.

Lemma 2.13. *For any* $t \in [\frac{1}{2}, 1]$,

$$||\varphi_t - \varphi_1||_{C^0} \;\leq\; c(1 + \mathbf{J}_\omega(\varphi_t - \varphi_1)), \tag{2.27}$$

where c *is a uniform constant.*

Proof. Both ω and $\omega_t = \omega_{\varphi_t - \varphi_1}$ have positive Ricci curvature $\geq 1/2$. So there are uniform bounds on the Sobolev constants of both metrics. Then (2.27) follows from the standard Moser iteration. □

Using (2.27) and applying Theorem 2.11, we get

$$\mathbf{F}_\omega(\varphi_t - \varphi_1) \;\geq\; c'\mathbf{J}_\omega(\varphi_t - \varphi_1)^\epsilon - C. \tag{2.28}$$

Then we proceed as before

$$\begin{aligned} \mathbf{F}_\omega(\varphi) &= -\mathbf{F}_{\omega'}(\varphi_1) \qquad\qquad (2.29)\\ &\geq (1-t)\,(\mathbf{I}_{\omega'}(\varphi_t) - \mathbf{J}_{\omega'}(\varphi_t))\\ &\geq (1-t)\,(\mathbf{I}_{\omega'}(\varphi_1) - \mathbf{J}_{\omega'}(\varphi_1)) - c(1-t)||\varphi_t - \varphi_1||_{C^0}\\ &= (1-t)\mathbf{J}_\omega(\varphi) - c(1-t)||\varphi_t - \varphi_1||_{C^0}\\ &\geq (1-t)\mathbf{J}_\omega(\varphi) - c''(1-t)\mathbf{J}_\omega(\varphi_t - \varphi_1). \end{aligned}$$

If $\mathbf{J}_\omega(\varphi_t - \varphi_1)$ is uniformly bounded for $t \in [1/2, 1]$, then we simply take $t = 1/2$ in the above inequality. Otherwise, we first note that (2.26) and (2.28) imply

$$c'\mathbf{J}_\omega(\varphi_t - \varphi_1)^\epsilon - C \;\leq\; (1-t)\,\mathbf{J}_\omega(\varphi).$$

Hence, we can choose $t \in [1/2, 1]$ such that

$$(1-t)\mathbf{J}_\omega(\varphi)^{1-\epsilon} = 1.$$

Then we have proved the following result due to Tian-Zhu:

Theorem 2.14. ([TZ97]) *Let (M, ω) be a Kähler-Einstein manifold with* $\mathrm{Ric}(\omega) = \omega$ *and without any non-trivial holomorphic vector fields. Then there are $C > 0$ and $\epsilon \in (0, 1)$ such that for any $\varphi \in P(M, \omega)$, we have*

$$\mathbf{F}_\omega(\varphi) \;\geq\; \mathbf{J}_\omega(\varphi)^\epsilon - C. \tag{2.30}$$

This gives an exact converse to Theorem 2.6 in the case that M does not have any non-trivial holomorphic fields. It is possible to remove this condition on holomorphic fields. More arguments are needed and will be discussed elsewhere.

Inspired by the above discussions, we can propose the following conjecture in [Ti97]. This is a natural extension of the famous Moser-Trudinger-Onofri inequality on S^2.

Conjecture 2.15. *Let (M, ω) be a Kähler-Einstein manifold with* $\mathrm{Ric}(\omega) = \omega$. *Then there are constants $\eta > 0$ and $C > 0$ such that for any $\varphi \in P(M, \omega)$ which is perpendicular to the kernel of $\Delta_\omega + 1$ (possibly trivial),*

$$\mathbf{F}_\omega(\varphi) \;\geq\; \eta\,\mathbf{J}_\omega(\varphi) - C. \tag{2.31}$$

When $\eta(M) = \{0\}$, this conjecture was verified by Phong et al [PSSW06] following arguments in [Ti97] and [TZ97]. For the readers' convenience, we reproduce their arguments here. Recall that (2.26) and (2.27) yield for $t \geq 1/2$,

$$\mathbf{F}_\omega(\varphi_t - \varphi_1) \;\leq\; C'(1-t)\mathbf{J}_\omega(\varphi_t - \varphi_1).$$

Combining this with (2.28), we get

$$c'\mathbf{J}_\omega(\varphi_t-\varphi_1)^\epsilon - C \;\le\; C''(1-t)\mathbf{J}_\omega(\varphi_t-\varphi_1).$$

If $C''\mathbf{J}_\omega(\varphi_{\frac{1}{2}}-\varphi_1)^{1-\epsilon}\le c'$, then (2.29) with $t=1/2$ yields (2.31) with $\eta=1/2$ Otherwise, there is some $t'\in(1/2,1)$ such that $C''(1-t')\mathbf{J}_\omega(\varphi_{t'}-\varphi_1)^{1-\epsilon}=c'/2$, then

$$c'\mathbf{J}_\omega(\varphi_{t'}-\varphi_1)^\epsilon \;\le\; 2C.$$

Consequently, we have $1-t'\ge\eta$ if η is sufficiently small, so we still deduce (2.31) from (2.29). therefore, the conjecture holds in the case that M has no non-trivial holomorphic fields.

3 Kähler-Einstein metrics on complex surfaces

This section concerns the existence of Kähler-Einstein metrics on complex surfaces with positive first Chern class, i.e., Del-Pezzo surfaces. This is completely solved in [Ti89]. Let us recall the main theorem in [Ti89].

Theorem 3.1. *Let M be a compact complex surface with positive first Chern class. Then M admits a Kähler-Einstein metric if and only if it has vanishing Futaki invariant.*

Of course, vanishing of the Futaki invariant is a necessary condition for the existence. In this section, we will first outline a proof of this theorem following arguments in [Ti89]. The main idea of my proof can be briefly described as follows: The main block of the proof is to prove the existence of Kähler-Einstein metrics on a Del-Pezzo surface M with vanishing Futaki invariant. First we establish the existence of Kähler-Einstein metrics on certain complex surface in the moduli of complex structures on M. This is done by computing the α-invariant I introduced. It turns out that the moduli is connected and the subset of those which admit Kähler-Einstein metrics is open. To prove that such a subset is also closed, we need an a prior C^0-estimate for solutions of certain complex Monge-Ampere equation. We achieve it in two steps:

1. We establish a partial C^0-estimate. This is done by proving a compactness theorem and using the L^2-estimate for the $\bar\partial$-operators;

2. With the partial C^0-estimate, one can reduce the properness of $\mathbf{F}_\omega$ to the properness of the restriction of $\mathbf{F}_\omega$ to certain finite dimensional space of "algebraically" defined Kähler metrics.[6] We then develop a method of verifying this finite dimensional properness.

3.1 The α-invariant and its applications

First let us introduce the α-invariant and state its basic properties for a general Fano manifold. As before, (M,ω) is a compact Kähler manifold with ω

[6]In fact, this properness is equivalent to the K-stability condition.

representing $\pi c_1(M)$. The α-invariant $\alpha(M)$, which is introduced in [Ti87], is defined as follows:

$$\alpha(M) := \sup\{\alpha \mid \int_M e^{-\alpha(\varphi - \sup_M \varphi)}\,\omega^n \le C_\alpha, \forall \varphi \in P(M,\omega)\,\}, \tag{3.1}$$

where C_α denotes a constant depending only on α. If G is a maximal compact subgroup G in the automorphism group $\mathrm{Aut}(M)$, the we can choose ω to be G-invariant and define

$$\alpha_G(M) := \sup\{\alpha \mid \int_M e^{-\alpha(\varphi - \sup_M \varphi)}\,\omega^n \le C_\alpha, \forall \varphi \in P(M,\omega)\,\}, \tag{3.2}$$

where $P_G(M,\omega)$ denotes the set of all $\varphi \in P(M,\omega)$ which is G-invariant. It is not hard to prove that both $\alpha(M)$ and $\alpha_G(M)$ are independent of the choices of ω and G, so they are holomorphic invariants of M. Moreover, we always have $\alpha_G(M) \ge \alpha(M) > 0$. The following theorem is proved in [Ti87] and provides a useful tool for establishing the existence of Kähler-Einstein metrics on Fano manifolds.

Theorem 3.2. *A Fano manifold M of dimension n admits a Kähler-Einstein metric if $\alpha(M) > \frac{n}{n+1}$ or more weakly, $\alpha_G(M) > \frac{n}{n+1}$.*

In fact, under the assumption on $\alpha(M)$ (resp. $\alpha_G(M)$), one can show (cf. [Ti98]) that $\mathbf{F}_\omega$ is proper on $P(M,\omega)$ (resp. $P_G(M,\omega)$), so this theorem follows from Theorem 2.6 or its variant for G-invariant functions.

Now we apply Theorem 3.2 to Del-Pezzo surfaces. By the classification theory of complex surfaces, M is either $\mathbb{C}P^2$ or $S^2 \times S^2$ or the blow-up Σ_m of $\mathbb{C}P^2$ at m points ($1 \le m \le 8$) in general position. Here the general position means that no three points are collinear and no six points are on a common quadratic curve and there are no cubic curves which contain 7 points and the 8th point as a double point. This is equivalent to saying that $c_1(\Sigma_m) > 0$.

The first two surfaces are homogeneous and so have canonical Kähler-Einstein metrics. It is proved in [Fut83] that M has a Kähler-Einstein metrics only if its associated Futaki invariant vanishes. It was also known that the Futaki invariant of $M = \Sigma_m$ is nonzero if and only if $m = 1$ or 2. Therefore, it suffices to establish the existence of Kähler-Einstein metrics on Σ_m for $3 \le m \le 8$.

The following is the main result in [TY87] and proved by computing $\alpha_G(M)$ for a maximal compact subgroup G in $\mathrm{Aut}(M)$.

Theorem 3.3. *For m between 3 and 8, there is at least one blow-up $M = \Sigma_m$ of $\mathbb{C}P^2$ with $c_1(M) > 0$ which admits a Kähler-Einstein metric.*

What we did in [TY87] is to prove $\alpha_G(\Sigma_m) > 2/3$ for each $m \in [3,8]$ and certain configuration of blow-up points such that Σ_m has sufficiently many symmetries. Denote by $\mathcal{M}_m$ the moduli spaces of Σ_m. It consists of all possible m points in $\mathbb{C}P^2$ in general position. Here is a summary of our computation of $\alpha_G(M)$:

(1) There is only one complex surface $\Sigma_3 \in \mathcal{M}_3$ and $\alpha_G(\Sigma_3) \ge 1$;

(2) There is only one surface $\Sigma_4 \in \mathcal{M}_4$ and $\alpha_G(\Sigma_4) \geq 3/4$;

(3) Every surface $\Sigma_5 \in \mathcal{M}_5$ is a complete intersection of two quadrics in $\mathbb{C}P^4$. If two quadratic polynomials are given by $\sum_{i=0}^4 z_i^2 = 0$ and $\sum_{i=0}^4 \lambda_i z_i^2 = 0$, then $\alpha_G(\Sigma_5) \geq 1$. It was pointed out in [MaMu93] that this is always the case for smooth Σ_5;

(4) Every surface in $\mathcal{M}_6$ is a cubic surface in $\mathbb{C}P^3$. If Σ_6 is the Fermat surface, then $\alpha_G(\Sigma_6) \geq 1$;

(5) Every surface M in $\mathcal{M}_7$ is a double branch covering of $\mathbb{C}P^2$ along a quartic curve Q. Then $\alpha_G(M) \geq 3/4$ when Q is a quartic curve with certain finite symmetries [7];

(6) Certain Σ_8 in $\mathcal{M}_8$ with finite symmetries has $\alpha_G(\Sigma_8) \geq 5/6$.

Note that the complex structure on Σ_3 or Σ_4 is unique. When $m \geq 5$, there is a connected complex moduli of Σ_m of dimension $m-4$.

Since [TY87], many advances have been made on computing $\alpha(\Sigma_m)$. Let us mention a few results which are directly related to Theorem 3.3. First we note that each Σ_5 is a complete intersection of two quadrics in $\mathbb{C}P^4$. As Mabuchi-Mukai stated in [MaMu93], one can diagonalize those two quadratics simultaneously. Then the arguments in [TY87] can be used to show that $\alpha_G(\Sigma_5) \geq 1$. In this way, Mabuchi-Mukai gave a simplified proof of Theorem 3.1 in the case Σ_5. More recently, I. Cheltsov et al proved that $\alpha(\Sigma_m) \geq 2/3$ for all $m \geq 5$ and $\alpha(\Sigma_m) \geq 3/4$ for $m \geq 6$ except $m = 6$ and Σ_m has an Eckardt point [Ch07]. Hence, there is an alternative proof of Theorem 3.1 by using directly α-invariants for all Del-Pezzo surfaces except for those cubic surfaces Σ_6 with an Eckardt point. We refer the readers to Appendix A for more about cubic surfaces with an Eckardt point.

3.2 Compactness for Kähler-Einstein metrics

To prove Theorem 3.1, we may assume that $m = 5, 6, 7, 8$. One can show that $\mathcal{M}_m$ is connected. Now we can set up a new continuity method: Let $\mathcal{E}_m$ be the subset of all $M \in \mathcal{M}_m$ which admits a Kähler-Einstein metric. It follows from last subsection that $\mathcal{E}_m$ is nonempty. Choose a smooth family of Kähler metrics ω_τ on $M_\tau \in \mathcal{M}_m$ with Kähler class $\pi c_1(M_\tau)$ and $M_0 \in \mathcal{E}_m$, where $\tau \in [0,1]$. Then M_τ admits a Kähler-Einstein metric if and only if the following Monge-Ampère equation is solvable

$$(\omega_\tau + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi)^2 = e^{h_\tau - \varphi}\omega_\tau^2, \quad \omega_\tau + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi > 0 \text{ on } M_\tau, \tag{3.3}$$

[7] It was proved in [Ti98] that every surface in $\mathcal{M}_7$ has proper $\mathbf{F}_\omega$ on $P_G(M, \omega)$ for a maximal compact subgroup $G \subset \mathrm{Aut}(M)$, so it has a Kähler-Einstein metric. This gives a simpler proof of Theorem 3.1 for Σ_7.

where h_τ is determined by

$$\mathrm{Ric}(\omega_\tau) - \omega_\tau = \frac{\sqrt{-1}}{2}\partial\overline{\partial} h_\tau, \quad \int_{M_\tau} (e^{h_\tau} - 1)\omega_\tau^2 = 0.$$

Since $m \geq 5$, any surface M_τ does not have any nontrivial holomorphic vector fields. Then by using the standard Bochner trick, one can show that for any solution of φ of (3.3), the first non-zero eigenvalue of $\Delta_{\tau,\varphi}$ is strictly bigger than 1, where $\Delta_{\tau,\varphi}$ denotes the Laplacian of the metric $\omega_\tau + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi$. It follows that if (3.3) is solvable on M_τ, so is every $M_{\tau'}$ sufficiently close to M_τ. This is a simple application of the Implicit Function Theorem since the linearization of (3.3) is given by $\Delta_{\tau,\varphi} + 1$. It implies that the set I_m of $\tau \in [0,1]$ with $M_\tau \in \mathcal{E}_m$ is open, and consequently, $\mathcal{E}_m$ is open in $\mathcal{M}_m$. It remains to show that I_m is closed. For this, we need an a priori C^3-estimate on solutions of (3.3). As we explained in [Ti87] or at the end of Section 1, this C^3-estimate follows from an a priori C^0-estimate. In general, there does not exist such an estimate for an equation of the type like (3.3). The idea of [Ti89] is to derive a partial C^0-estimate and then use geometric information of underlying manifolds to check if the required C^0-estimate holds. Now let us recall the partial C^0-estimate.

Theorem 3.4. *[Ti89] There are two constants $c > 0$ and $l_0 > 0$ such that for any Kähler-Einstein surface (M, ω) with $\mathrm{Ric}(\omega) = \omega$, there is some $l \in [l_0, 2l_0]$ such that*

$$c^{-1} \geq \frac{1}{l}\sum_{i=0}^{N} ||S_i||^2 \geq c > 0, \tag{3.4}$$

where $\{S_i\}_{0\leq i\leq N}$ is any orthonormal basis of $H^0(M, K_M^{-l})$ with respect to the inner product induced by ω. [8]

Let us first explain why it implies a partial C^0-estimate[9]: Let $M = \mathfrak{M}_\tau$ and φ be a solution of (3.3). Write ω for $\omega_\tau + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi$. Choose a Hermitian metric $||\cdot||_\tau$ on K_M^{-1} such that its curvature form is ω_τ. This Hermitian metric and ω_τ induces an inner product on $H^0(M, K_M^{-l})$. We may choose $\{S_i\}$ in (3.4) such that $\{\mu_i S_i\}$ is an orthonormal basis of this inner product associated to ω_τ for some positive constants μ_i $(i = 0, \cdots, N)$. It follows from the Maximum principle

$$\varphi - \frac{1}{l}\log\left(\frac{\sum_{i=0}^N ||S_i||_\tau^2}{\sum_{i=0}^N ||S_i||^2}\right) = c', \tag{3.5}$$

where c' is some constant. Write $\sigma_i = \mu_i S_i$. We may arrange $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_N$ and put $\lambda_i = \mu_N/\mu_1$. Then $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_N = 1$, also it follows from Theorem 3.4

$$||\varphi - \sup_M \varphi - \frac{1}{l}\log(\sum_{i=0}^N \lambda_i^2 ||\sigma_i||_\tau^2)||_{C^0} \leq C, \tag{3.6}$$

[8] In [Ti89], (3.4) is actually proved for any $l = 6k > 0$. It was also conjectured that (3.4) holds for any l sufficiently large.

[9] This derivation from (3.4) to the partial C^0-estimate, i.e. (3.6), for solutions of (3.3) holds for any dimensions.

Note that C always denotes a uniform constant. In particular, $\varphi - \sup_M \varphi$ is bounded away from the zero locus of σ_N. So we have a partial C^0-estimate of $\varphi - \sup_M \varphi$.

Now we discuss how to prove Theorem 3.4. The key is the following compactness theorem for Kähler-Einstein metrics proved in [Ti89].

Theorem 3.5. *For any sequence of Kähler-Einstein surfaces (M_i, ω_i) with $M_i \in \mathcal{E}_m$ and $\mathrm{Ric}(\omega_i) = \omega_i$, there is a subsequence, for simplicity, still denoted by $\{(M_i, \omega_i)\}$, converging to a Kähler-Einstein orbifold $(M_\infty, \omega_\infty)$ in the Cheeger-Gromov topology satisfying:*

(1) The singularities are of the form U/Γ and the number of them is uniformly bounded, where $U \subset \mathbb{C}^2$ and Γ is a finite group of $U(2)$ with uniformly bounded order;

(2) For each $\ell > 0$, $H^0(M_i, K_{M_i}^{-\ell})$ converge to $H^0(M_\infty, K_{M_\infty}^{-\ell})$ in the following sense [10]: There are orthonormal bases $\{S_0^i, \cdots, S_N^i\}$ of $H^0(M_i, K_{M_i}^{-\ell})$ converging to an orthonormal basis of $H^0(M_\infty, K_{M_\infty}^{-\ell})$.

Remark 3.6. *A real version of this theorem without (2) was given in [An90] with an earlier and weaker version given in [Na88]. We note that (2) is crucial in our deriving the partial C^0-estimate.*

Remark 3.7. *It was conjectured in [Ti89] that those quotient singularities are all rational double points, that is, the uniformization group Γ in (1) of Theorem 3.5 lies in $SU(2)$. Indeed, a result of Mabuchi-Mukai implies this for $m = 5$ [MaMu93]. In fact, we know more about Γ in [Ti89]: Choose $p_i \in M_i$ which converge to a singularity $p_\infty \in M_\infty$ which has a neighborhood of the form U/Γ, then there are open neighborhoods V_i of p_i in M_i which is a finite quotient of a smooth deformation of U/Γ_0 for some $\Gamma_0 \subset SU(2)$. It follows that either $\Gamma \subset SU(2)$ or Γ is cyclic. Using the volume comparison and the bound $c_1(M)^2([M]) = 9 - m$, one can show that the order $\Gamma/\Gamma_0 \leq 6$.*

It follows that the moduli space $\mathcal{E}_m$ of Kähler-Einstein surfaces with positive scalar curvature can be compactified by adding Kähler-Einstein orbifolds with isolated singularities described in the above theorem. Clearly, Theorem 3.4 follows from this: Let ℓ is a product of orders of all finite groups which appear in the quotient singularities in the compactification, then for any orbifold $(M_\infty, \omega_\infty)$ in the boundary of $\mathcal{E}_m$ and any $p \in M_\infty$, there is at least one section $S \in H^0(M_\infty, K_{M_\infty}^{-\ell})$ with its L^2-norm equal to 1 such that $||S(p)|| \geq c$ for a uniform constant $c = c(\ell)$. Thus we can easily deduce (3.4).

To prove (1) in Theorem 3.5, we recall that for any Kähler-Einstein surface (M, ω) with $M \in \mathcal{E}_m$ and $\mathrm{Ric}(\omega) = \omega$, we have a fixed volume:

$$\mathrm{Vol}(M, \omega) := \int_M \omega^n = 2^n \pi^n c_1(M)^n([M]),$$

[10] This can be regarded as a refined version of the flatness for a family of complex manifolds.

and by the Gauss-Bonnet-Chern formula,

$$\frac{1}{8\pi^2}\int_M ||Rm(\omega)||^2_\omega \omega^n \;=\; c_2(M)([M]),$$

where $Rm(\cdot)$ denotes the curvature tenor and $c_2(M)$ denotes the second Chern class. This implies that the L^2-norm of the curvature is uniformly bounded. Moreover, by the Meyer theorem, the diameter of (M,ω) is uniformly bounded, so we have the uniform Sobolev inequality. In [Ti89], using this Sobolev bound and adopting Uhlenbeck's estimate for Yang-Mills connections, I gave a curvature estimate in terms of the local L^2 norm of curvature. This enables us to bound curvature outside finitely many points of M. Then we can deduce (1) from this curvature estimate and a removable singularity theorem (cf. [Ti89]).

To prove (2) in Theorem 3.5, we use Hömander's L^2-estimate[11]. First we notice that using the Bochner identity and the Moser iteration, one can bound $||S||_{\omega_i}$ for any section $S \in H^0(M_i, K_{M_i}^{-\ell})$ with its L^2-norm equal to 1. Therefore, by taking a subsequence if necessary, we may assume that $\{S_a^i\}_{0\le a\le N}$ converge to a sub-basis of $H^0(M_\infty, K^\ell_{M_\infty})$. So it suffices to show that any section $S \in H^0(M_\infty, K^\ell_{M_\infty})$ is the limit of a sequence of sections on M_i. This is done by using the fact that the singular set is of codimension at least 4 and the L^2-estimate.

3.3 Analytic stability in finite dimensions

We adopt the notations from the last subsection. In the final stage of proving Theorem 3.1, we will prove that for any solution φ of (3.3),

$$\mathbf{F}_{\omega_\tau}(\varphi) \ge \epsilon \mathbf{J}_{\omega_\tau}(\varphi) - C, \tag{3.7}$$

where $\epsilon > 0$ is independent of $\tau \in I_m$, φ and φ solves (3.3).[12] Note that C always denotes a uniform constant in this section. By (2.23), we have

$$\mathbf{F}_{\omega_\tau}(\varphi) \le 0.$$

It follows from (3.7) that $\mathbf{J}_{\omega_\tau}(\varphi)$ is uniformly bounded and consequently, $||\varphi||_{C^{2,\frac{1}{2}}}$ is uniformly bounded. It follows that I_m is closed in [0,1]. Since $\{M_\tau\}$ can be any family in $\mathcal{M}_m$ with $M_0 \in \mathcal{E}_m$. we have $\mathcal{E}_m = \mathcal{M}_m$, thus Theorem 3.1 is proved.

Now let us prove (3.7) for M_τ. Since φ is a solution of (3.3), we have

$$\begin{aligned}\mathbf{F}_{\omega_\tau}(\varphi) &= \frac{\sqrt{-1}}{3V}\int_M \partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\tau + \frac{\sqrt{-1}}{6V}\int_M \partial\varphi\wedge\overline{\partial}\varphi\wedge\omega - \frac{1}{V}\int_M \varphi\omega_\tau^2 \\ &= -\frac{1}{3V}\int_M \varphi\omega^2 - \frac{2}{3V}\int_M \varphi\omega_\tau^2 + \frac{\sqrt{-1}}{6V}\int_M \partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\tau. \end{aligned} \tag{3.8}$$

[11]We should point out that arguments in proving (2) hold for all dimensions. More details will be given in the next section.
[12]This means that $\mathbf{F}_{\omega_\tau}$ is proper along the solutions of (3.3).

Since $\mathbf{F}_{\omega_\tau}(\varphi) \le 0$, we deduce from this

$$-\inf_M \varphi \;\le\; -\frac{1}{V}\int_M \varphi\omega^2 \;\le\; \frac{2}{V}\int \varphi\omega_\tau^2. \tag{3.9}$$

However, by using the Green function, we have (see [Ti87]):

$$\sup_M \varphi \;\le\; \frac{1}{V}\int_M \varphi\omega_\tau^2 + C. \tag{3.10}$$

Hence, we get

$$-\inf_M \varphi \;\le\; 2\sup_M \varphi + C. \tag{3.11}$$

We will use certain finite dimensional version of $\alpha(M_\tau)$ to derive (3.7). Recall the definition of $\alpha_{\ell,k}(M_\tau)$ $(1 \le k \le N+1)$ (see Appendix A for more details): Choose any Hermitian metric h on $K_M^{-\ell}$ whose curvature form $\omega' > 0$, where $M = M_\tau$. It induces a Hermitian inner product $(\cdot,\cdot)_h$ on $H^0(M, K_M^{-\ell})$. We define (cf. (5.1))

$$\alpha_{\ell,k}(M) = \sup\{\alpha \,|\, \int_M \frac{\omega'^2}{\left(\sum_{i=1}^k ||S_i||_h^2\right)^{\frac{\alpha}{\ell}}} \le C_\alpha,\ \forall \text{ orthonormal subbasis } \{S_i\}\}.$$

Here C_α denotes a uniform constant depending only on α.

We will derive (3.7) by using $\alpha_{\ell,1}(M)$ and $\alpha_{\ell,2}(M)$. The following lemma is a refined version of a corresponding lemma proved in [Ti89]. We make the constants more precise, but the lemma in [Ti89] is sufficient for proving (3.7).

Lemma 3.8. *Let M be a smooth Del Pezzo surface obtained by blowing up $\mathbb{C}P^2$ at m points in general position ($5 \le m \le 8$). Assume that for some $\ell > 0$, the partial C^0-estimate* (3.4) *holds and*

$$\frac{1}{\alpha_{\ell,1}(M)} + \frac{1}{\alpha_{\ell,2}(M)} \;<\; 3,$$

then (3.7) *holds.*

Proof. Let us give an outline of its proof. First we recall that (3.4) implies

$$||\varphi - \sup_M \varphi - \frac{1}{\ell}\log(\sum_{i=0}^N \lambda_i^2 ||\sigma_i||_\tau^2)||_{C^0} \le C,$$

where $0 < \lambda_0 \le \cdots \le \lambda_{N-1} \le \lambda_N = 1$ are given in (3.6).

Put

$$\psi \;=\; \frac{1}{\ell}\log\left(\sum_{i=0}^N \lambda_i^2 ||\sigma_i||_\tau^2\right).$$

Using the concavity of logarithm and (3.3), for any $\alpha < \alpha_{\ell,2}(M)$, we can have

$$\begin{aligned}
& \alpha \sup_M \varphi + \frac{1-\alpha}{V}\int_M \varphi\,\omega^2 \\
\le\ & \log\left(\frac{1}{V}\int_M e^{\alpha sup_M\varphi+(1-\alpha)\varphi} e^{h_\tau-\varphi}\omega_\tau^2\right) \\
\le\ & \log\left(\frac{1}{V}\int_M e^{-\alpha(\varphi-\sup_M\varphi)}\omega_\tau^2\right) + \sup_M h_\tau \\
\le\ & \log\left(\frac{1}{V}\int_M e^{-\frac{\alpha}{\ell}(\log\lambda_{N-1}^2+\log(||\sigma_{N-1}||_\tau^2+||\sigma_N||_\tau^2))}\,\omega_\tau^2\right) + C' \\
\le\ & -\frac{\alpha}{\ell}\log\lambda_{N-1}^2 + C'_\alpha. \qquad (3.12)
\end{aligned}$$

A direct computation shows that for any $\delta > 0$, there is a uniform C_δ such that

$$\frac{\sqrt{-1}}{2V}\int_M \partial\psi\wedge\overline{\partial}\psi\wedge\omega_\tau \ \ge\ -\frac{1-\delta}{\ell}\log\lambda_{N-1} - C_\delta. \qquad (3.13)$$

Note that

$$\psi = \frac{1}{\ell}\log(\sum_{i=0}^N \lambda_i^2||\sigma_i||_\tau^2).$$

Plugging (3.13) into (3.8) and using (3.10), we have

$$\mathbf{F}_{\omega_\tau}(\varphi) \ \ge\ -\frac{1}{3V}\int_M \varphi\omega^2 - \frac{2}{3}\sup_M\varphi - \frac{1-\delta}{3\ell}\log\lambda_{N-1}^2 - C'_\delta.$$

Combining this with the estimate (3.12), we get

$$\mathbf{F}_{\omega_\tau}(\varphi) \ \ge\ -\left(\frac{1}{3} - \frac{(1-\delta)(1-\alpha)}{3\alpha}\right)\frac{1}{V}\int_M \varphi\omega^2 - \frac{1+\delta}{3}\sup_M\varphi - \tilde{C}_\delta. \qquad (3.14)$$

On the other hand, for any $\beta < \alpha_{\ell,1}(M)$, using the arguments in deriving (3.12), we get

$$\beta\sup_M\varphi + \frac{1-\beta}{V}\int_M \varphi\,\omega^2 \ \le\ C_\beta.$$

Combining this with (3.14), we have

$$\mathbf{F}_{\omega_\tau}(\varphi) \ \ge\ \frac{\beta}{3(1-\beta)}\left(3 - \frac{1-\delta}{\alpha} - \frac{1+\delta}{\beta}\right)\sup_M\varphi - C_1. \qquad (3.15)$$

By our assumption, we can choose α, β and δ such that the coefficient of $\sup_M\varphi$ in (3.15) is positive. □

Remark 3.9. *One can show that* (3.7) *is equivalent to the following finite-dimensional problem:*

$$\mathbf{F}_{\omega_\tau}(\psi) \ \ge\ \epsilon\mathbf{J}_{\omega_\tau}(\psi) - C'. \qquad (3.16)$$

Corollary 3.10. *For any* $\tau \bar{I}_m$*, if* $\alpha_{\ell,1}(M_\tau) \geq 2/3$ *and* $\alpha_{\ell,2}(M_\tau) > 2/3$*, then* M_τ *admits a Kähler-Einstein metric and consequently,* I_m *is closed.*

Proof. Let $\tau_i \in I_m$ converging to τ. Then we have the partial C^0-estimate (3.4) for $(M_{\tau_i}, \omega_{\tau_i})$. Using the above lemma and our assumptions, we see that (3.7) holds for $(M_{\tau_i}, \omega_{\tau_i})$. Because of Theorem 5.5, the constants in (3.7) can be made uniform on τ_i. Hence, M_τ admits a Kähler-Einstein metric. □

To finish the proof of Theorem 3.1, we only need to prove that $\alpha_{\ell,1}(M_\tau) \geq 2/3$ and $\alpha_{\ell,2}(M_\tau) > 2/3$ for some ℓ such that (3.4) holds for $\tau \in I_m$. In [Ti89], I prove (3.4) for $\ell = 6k$ and verify the lower bounds on $\alpha_{6,1}(M_\tau)$ and $\alpha_{6,2}(M_\tau)$. Then Theorem 3.1 follows.

Recently, in [Shi09], Yalong Shi proves that $\alpha_{\ell,2}(M) > 2/3$ for any smooth cubic surface $M \subset \mathbb{C}P^3$ (also see Theorem 5.2). Combining this with the results of I. Cheltsov ([Ch07] and [Ch08], also Theorem 5.1), he gives a simpler and elegant proof for Theorem 3.1.

Remark 3.11. *It will be an interesting problem to study when there is a Kähler-Einstein orbifold metric on complex surfaces with isolated quotient singularities. Part of the proof of Theorem 3.1 can be adapted for the case of orbifolds, but there are substantial new difficulties due to the presence of singularities.*

3.4 An approach by Kähler-Ricci flow

In this subsection, we show how to adapt the above arguments in last two subsections to give an alternative proof of Theorem 3.1 by using the Kähler-Ricci flow. The route for the proof is identical. A detailed proof has been recently produced by Chen-Wang [ChWa09].

First we collect some general facts for the Kähler-Ricci flow on any Fano manifold. Consider the Kähler-Ricci flow:

$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)) + \omega(t), \qquad \omega(0) = \omega_0. \tag{3.17}$$

There are φ such that $\omega = \omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi$ and

$$\frac{\partial \varphi}{\partial t} = \log\left(\frac{(\omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi)^n}{\omega_0^n}\right) + h_0 + \varphi, \quad \varphi(0) = 0, \tag{3.18}$$

where $h_0 = h_{\omega_0}$. It is proved in [Cao86] that (3.18), and consequently (3.17), has a global solution $\varphi(t)$ for $t \geq 0$. Moreover, if $||\varphi(t)||_{C^0}$ is uniformly bounded, then there is a uniform bound on $||\varphi(t)||_{C^3}$ and $\omega(t)$ converges to a Kähler-Einstein metric on M. It implies Theorem 3.1. Therefore, as for the continuity method, we need an a prior C^0-estimate on any solution of (3.18).

Along the flow (3.18), we have

$$\frac{\partial}{\partial t}\mathbf{F}_{\omega_0}(\varphi) = -\int_M \frac{\partial \varphi}{\partial t}\left(e^{\frac{\partial \varphi}{\partial t}} - 1\right) e^{h_0 - \varphi}\omega_0^n \leq 0. \tag{3.19}$$

It implies that $\mathbf{F}_{\omega_0}$ is non-increasing along (3.18). In particular, if $\mathbf{F}_{\omega_0}$ is proper, then we can bound $\mathbf{J}_{\omega_0}(\varphi)$ uniformly. If we have the uniform Sobolev inequality for $\omega(t)$, then $||\varphi||_{C^0}$ is uniformly bounded. The Sobolev inequality along the Kähler-Ricci flow has been established by Q. Zhang and R.G. Ye based on Perelman's fundamental works on Ricci flow (see [QZ07], [Ye07]). Hence, as before, the key is to establish the properness of $\mathbf{F}_{\omega_0}$.

Now we assume that $M = \Sigma_m$ is a Del-Pezzo surface. We may assume $m \in \{5,6,7,8\}$, since for $m = 3,4$ there is a unique Σ_m in the moduli space which is Kähler-Einstein by Theorems 3.2 and 3.3. Let ω_0 be a Kähler metric with $\pi c_1(M)$ as its Kähler class. In [Ch07], Cheltsov proved $\alpha(M) > 2/3$ unless $m = 5$ or $m = 6$ and M has an Eckardt point. If $m = 5$ and ω_0 is invariant under a maximal compact subgroup G, then $\alpha_G(M) > 2/3$. Actually, in [Ru08], Rubinstein proves that the Ricci flow (3.17) converges to a Kähler-Einstein metric if $\alpha(M) > 2/3$ or if ω_0 is G-invariant and $\alpha_G(M) > 2/3$.[13] Hence, we have a Ricci flow proof for Theorem 3.1 in those cases. An alternative approach to this is to use the properness of $\mathbf{F}_{\omega_0}$ (cf. [ChWa09]): $\mathbf{F}_{\omega_0}$ is proper in the case of $m > 6$ or $m = 6$ and M does not admit any Eckardt points, or $\mathbf{F}_{\omega_0}$ is proper on $P_G(M,\omega_0)$ if $m = 5$ and ω_0 is invariant under a maximal compact subgroup G.

But it remains to prove the case of $M = \Sigma_6$ with an Eckardt point. For this, we follow the route in [Ti89] or the arguments in last two subsections. But one needs to work out a new compactness theorem analogous to Theorem 3.5. The following is proved in [ChWa09].

Theorem 3.12. *For any sequence $\{t_i\}$ with $\lim t_i = \infty$, by taking a subsequence if necessary, $(M, \omega(t_i)$ converge to a Kähler-Einstein orbifold $(M_\infty, \omega_\infty)$ in the Cheeger-Gromov topology satisfying:*

(1) The singularities are of the form U/Γ and the number of them is uniformly bounded, where $U \subset \mathbb{C}^2$ and Γ is a finite group of $U(2)$ with uniformly bounded order;

(2) For each $\ell > 0$, $H^0(M, K_M^{-\ell})$ converge to $H^0(M_\infty, K_{M_\infty}^{-\ell})$ in the following sense There are orthonormal bases $\{S_0^i, \cdots, S_N^i\}$ of $H^0(M, K_M^{-\ell})$ converging to an orthonormal basis of $H^0(M_\infty, K_{M_\infty}^{-\ell})$. Here the inner product is induced by a Hermitian metric on K_M^{-1} with $\omega(t_i)$ as its curvature form.

To say a few words about its proof, we remark that in place of Kähler-Einstein condition, one can use a result of Perelman: The scalar curvature of $\omega(t)$ is uniformly bounded. Moreover, the Sobolev constants are uniformly bounded. Hence, as before, one can develop the curvature estimate and use the L^2-estimate for the $\bar\partial$-operator.

[13] Rubinstein's result holds for any dimensions and is a Ricci flow version of the main theorem in [Ti87].

Theorem 3.12 yields the partial C^0-estimate for $\varphi(t)$:

$$||\varphi(t) - \sup_M \varphi(t) - \frac{1}{\ell}\log(\sum_{i=0}^N \lambda_i(t)^2||\sigma_i(t)||_0^2)||_{C^0} \le C.$$

Next one proves the properness of $\mathbf{F}_{\omega_0}$ for the above $\frac{1}{\ell}\log(\sum_{i=0}^N \lambda_i(t)^2||\sigma_i(t)||_0^2)$. As before, it can be done by using $\alpha_{\ell,1}(M) \ge 2/3$ and $\alpha_{\ell,2}(M) > 2/3$ as we did in the last subsection. Hence a Ricci flow proof of Theorem 3.1 can be completed. We refer the readers to [ChWa09] for details.

4 What about higher dimensions

In this section, we discuss the existence of Kähler-Einstein metrics on Fano manifolds in higher dimensions. This problem faces new challenges:

1. The vanishing of Futaki invariants is no longer sufficient for the existence, in fact, there is a Fano 3-fold without any hon-trivial holomorphic vector fields and which does not admit any Kähler-Einstein metrics, either (cf. [Ti97]). Hence, a geometric condition needs to be found. It was already speculated in the late 80's that the correct condition would be in terms of a certain stability of the underlying manifold in the sense of the Geometric Invariant Theory. Now it seems to be apparent that the correct condition is the K-stability first introduced in [Ti97] and reformulated in [Do02] in a purely algebraic way;

2. It is much harder to establish the partial C^0-estimate (3.5) in higher dimensions. The difficulty arises from the fact that there may be more possible singularities in compactifying the moduli of Kähler-Einstein manifolds in higher dimensions, or more generally, in the Gromov-Hausdorff limits of Kähler spaces with Ricci curvature bounded from below by a positive number. Motivated by this, a compactness theorem was proved by Cheeger-Colding-Tian fifteen years ago. We will discuss this compactness theorem and how it can be applied to studying the existence problem and what remains to be done.

There are four subsections in this section. In the first, we discuss the K-stability. In the second, we discuss the partial C^0-estimate. In the third, we show how the K-stability implies the existence under the assumption of the partial C^0-estimate. In the last, we discuss what is known on the Kähler-Ricci flow.

4.1 The K-stability

In this subsection, we discuss the K-stability: First the definition from [Ti97] by using the generalized Futaki invariants, then a purely algebraic definition due to Donaldson [Do02]. We will also discuss how the K-stability is related to the existence of Kähler-Einstein metrics.

First let us recall the definition of the Futaki invariant [Fut83]: Let M be any Fano manifold and ω be a Kähler metric with $\pi c_1(M)$ as its Kähler class, for any holomorphic vector field X on M, Futaki defined

$$f_M(X) \;=\; \int_M X(h_\omega)\,\omega^n. \tag{4.1}$$

Note that $\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2}\partial\bar\partial h_\omega$. Futaki proved in [Fut83] that $f_M(X)$ is independent of the choice of ω, so it is a holomorphic invariant. Later, Bando, Calabi and Futaki observed that the invariant can be defined for any polarized Kähler manifold (M, Ω): If ω is any Kähler metric with cohomology class Ω and h_ω satisfies $s(\omega) - \underline{s} = \Delta_\omega h_\omega$, where $s(\omega)$ denotes the scalar curvature and $\underline{s}$ is its average, then one can still show that the integral in (4.3) depends only on M, Ω and X. Hence, we have an invariant, denoted by $f_{M,\Omega}(X)$. The Futaki invariant is an obstruction to the existence of Kähler-Einstein metrics on Fano manifolds, but its vanishing does not assure the existence as shown in [Ti97]. It motivates us to introduce the K-stability. For this, we need to extend the Futaki invariant to singular varieties. In [DT92], it was done for normal Fano varieties. It turns out that the arguments also apply to more general cases. To be more convincing, we reformulate the definition of $f_{M,\Omega}(X)$. Note that

$$i_X\omega = \frac{\sqrt{-1}}{2}(H_X + \bar\partial\theta_X),$$

where H_X is a parallel (0,1)-form. One can show

$$f_{M,\Omega}(X) \;=\; -n\int_M \theta_X\left(\mathrm{Ric}(\omega) - \frac{\underline{s}}{n}\omega\right)\wedge\omega^{n-1}. \tag{4.2}$$

In particular, parallel (0,1)-forms have no effect in $f_{M,\Omega}(X)$. For simplicity, we will drop it and consider only those fields X with $H_X = 0$.

The formula (4.2) allows to extend the Futaki invariant to singular varieties. For our purpose, we only need to define the invariant for any polarized normal variety (M, L) and the following vector field: Assume that there is an embedding $M \subset \mathbb{C}P^N$ such that $L^\ell = \mathcal{O}(1)|_M$ and X is a holomorphic vector field of $\mathbb{C}P^N$ restricted to M, then there is a smooth function θ_X on $\mathbb{C}P^N$ such that

$$i_X\omega = \frac{\sqrt{-1}}{2}\bar\partial\theta_X.$$

where ω is the restriction of $\frac{1}{\ell}\omega_{FS}$ to the regular part of M.[14] It is well-known that $\mathrm{Ric}(\omega)$ is bounded from above and its trace, the scalar curvature, is L^1-bounded. Therefore, the integral in (4.2) is still meaningful. Moreover, one can prove that it is independent of the choice of ω, so it defines the generalized Futaki invariant $f_{M,L}(X)$ for polarized normal variety (M, L).[15] The proof is not very hard and the details will be skipped here.

[14]We will always denote ω_{FS} the Fubini-Study metric on $\mathbb{C}P^N$ in this paper.

[15]The integral (4.2) also makes sense for non-normal varieties, but the correct Futaki invariant has contributions from singularity of codimension 1 since we want to preserve certain natural continuity on any family of varieties.

The formula (4.2) can be used further as in [DT92]. Let (M, L) be a polarized manifold. By the Kodaira embedding theorem, for ℓ sufficiently large, a basis of $H^0(M, L^\ell)$ gives an embedding $\phi_\ell : M \mapsto \mathbb{C}P^N$, where $N = \dim_{\mathbb{C}} H^0(M, L^\ell) - 1$. Any other basis gives an embedding of the form $\sigma \cdot \phi_\ell$, where $\sigma \in G = \mathbf{SL}(N + 1, \mathbb{C})$. We fix such an embedding and consider the action of G on M.

Fix a Hermitian metric $|| \cdot ||$ on L such that its curvature form ω is a Kähler metric. Then for any $\sigma \in G$, there is a unique function φ_σ such that

$$\phi_\ell^* \sigma^* (|| \cdot ||_{FS}^{\frac{2}{\ell}}) \;=\; e^{-\varphi_\sigma} || \cdot ||^2, \tag{4.3}$$

where $|| \cdot ||_{FS}$ is a Hermitian metric on the hyperplane bundle over $\mathbb{C}P^N$ whose curvature form is ω_{FS}.

It is known that for any algebraic subgroup $G_0 = \{\sigma_t\}_{t \in \mathbb{C}^*}$ of $\mathbf{SL}(N + 1, \mathbb{C})$, there is a unique limiting cycle

$$M_0 = \lim_{t \to 0} \sigma_t(M) \subset \mathbb{C}P^N.$$

Let X be the holomorphic vector field whose real part generates the action by $\sigma(e^{-s})$, that is,

$$\frac{d\sigma(e^{-s})}{ds} \;=\; \mathrm{Re}(X)(\sigma(e^{-s})).$$

The following lemma is proved in [DT92] for Fano manifolds. However, the proof for the general case is identical.

Lemma 4.1. *Assume that G_0 is as above and M_0 is a normal variety. Then we have*

$$\lim_{s \to \infty} \frac{d}{ds} \left(\mathbf{T}_\omega(\varphi_{\sigma(e^{-s})}) \right) \;=\; \mathrm{Re}(F_{M_0, \Omega}(X)), \tag{4.4}$$

where $\Omega = \frac{1}{\ell}[\omega_{FS}]|_{M_0}$.

Proof. We follow [DT92] to prove this lemma. Since $\mathrm{Im}(X)$ is a Killing field for ω_{FS}, there is a real function θ such that

$$\frac{1}{\ell} L_X \omega_{FS} \;=\; \frac{\sqrt{-1}}{2} \partial \bar{\partial} \theta.$$

Differentiating in s the identity

$$\frac{1}{\ell} \sigma(e^{-s})^* \omega_{FS} \;=\; \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi_{\sigma(e^{-s})},$$

we get (possibly up to a constant)

$$\frac{d}{ds} \left(\varphi_{\sigma(e^{-s})} \right) (\sigma(e^s)(x)) \;=\; \theta(x), \quad \forall x \in \sigma(e^{-s})(M).$$

Then by the definition (2.9), we can deduce

$$\frac{d\mathbf{T}_\omega(\varphi_{\sigma(e^{-s})})}{ds} \;=\; -\frac{1}{V} \int_{\sigma(e^{-s})(M)} \theta \left(s(\omega_{\varphi_{\sigma(e^{-s})}}) - \underline{s} \right) \omega^n_{\varphi_{\sigma(e^{-s})}}.$$

There is a unique f_s on $\sigma(e^{-s})(M)$ such that

$$s(\omega_{\varphi_{\sigma(e^{-s})}}) - \underline{s} \,=\, \Delta_s f_s \quad\text{and}\quad \inf f_s = 1,$$

where Δ_s is the Laplacian of the metric $\frac{1}{\ell}\omega_{FS}$ restricted to $\sigma(e^{-s})(M)$. Hence, we have

$$\frac{d\mathbf{T}_\omega(\varphi_{\sigma(e^{-s})})}{ds} \;=\; \frac{1}{V}\int_{\sigma(e^{-s})(M)} \nabla\theta\cdot\nabla f_s\,\omega^n_{\varphi_{\sigma(e^{-s})}}. \tag{4.5}$$

Since M_0 is a normal variety and $s(\omega_{\varphi_{\sigma(e^{-s})}})$ is uniformly bounded from above, we can prove that f_s converge to f_∞ outside the singular set of M_0 as $s\to\infty$. Moreover, we have uniformly bounded Sobolev constant C_s for $(\sigma(e^{-s})(M), \frac{1}{\ell}\omega_{FS})$, so we have

$$\begin{aligned}
&\left(\int_{\sigma(e^{-s})(M)} f_s^{\frac{2n}{2n-1}}\,\omega^n_{\varphi_{\sigma(e^{-s})}}\right)^{\frac{n-1}{2n}} \qquad\qquad (4.6)\\
\le\;& C_s\int_{\sigma(e^{-s})(M)} |\nabla f_s^{\frac{1}{2}\left(1-\frac{1}{2n-1}\right)}|^2\,\omega^n_{\varphi_{\sigma(e^{-s})}}\\
=\;& \frac{n^2C_s}{2n-1}\int_{\sigma(e^{-s})(M)} f_s^{-\frac{1}{2n-1}}\,\Delta_s f_s\,\omega^n_{\varphi_{\sigma(e^{-s})}}\\
=\;& \frac{n^2C_s}{2n-1}\int_{\sigma(e^{-s})(M)} f_s^{-\frac{1}{2n-1}}\,(s(\omega_{\varphi_{\sigma(e^{-s})}})-\underline{s})\,\omega^n_{\varphi_{\sigma(e^{-s})}}.
\end{aligned}$$

The last integral is uniformly bounded. It follows that

$$\begin{aligned}
&\left(\int_{\sigma(e^{-s})(M)} |\nabla f_s|\,\omega^n_{\varphi_{\sigma(e^{-s})}}\right)^2\\
\le\;& \int_{\sigma(e^{-s})(M)} f_s^{\frac{2n}{2n-1}}\,\omega^n_{\varphi_{\sigma(e^{-s})}}\int_{\sigma(e^{-s})(M)} f_s^{-\frac{2n}{2n-1}}|\nabla f_s|^2|^2\,\omega^n_{\varphi_{\sigma(e^{-s})}}.
\end{aligned}$$

Combining this with (4.5) and (4.6), we can easily deduce (4.4). Here we have used the fact that θ has bounded gradient. □

We will denote by $\mathbf{w}(M,L,G_0)$ the generalized Futaki invariant in (4.4) and call it the weight of G_0 associated to (M,L) (abbreviated as $\mathbf{w}(G_0)$ if no confusion). In fact, the limit in (4.4) exists without any assumption on M_0, so the weight $\mathbf{w}(M,L,G_0)$ for any G_0 can be defined. This is proved in [PT04] when M_0 has no components of multiplicity greater than one or in [Paul08] for the general case.

Now we are ready to define the K-stability introduced in [Ti97]. Because our main topic in this paper is on Kähler-Einstein metrics with positive scalar curvature, we first assume that $M\subset\mathbb{C}P^N$ is a Fano manifold and $L=K_M^{-1}$ even though the definition for the general cases is very similar. We always denote by G_0 an one-parameter subgroup of $\mathbf{SL}(N+1)$. Let M_0 be the corresponding limit cycle.

Definition 4.2. *We say that M is K-semistable with respect to $K_M^{-\ell}$ if $\mathbf{w}(G_0) \geq 0$ for any $G_0 \subset \mathbf{SL}(N+1)$ such that the corresponding M_0 is a normal variety. We say that M is K-stable with respect to $K_M^{-\ell}$ if it is K-semistable and $\mathbf{w}(G_0) > 0$ for any $G_0 \subset \mathbf{SL}(N+1)$ such that the corresponding M_0 is a normal variety and not biholomorphic to M.*[16]

For $G_0 \subset \mathbf{SL}(N+1)$, we can choose homogeneous coordinates $z_0, \cdots, z_N$ for $\mathbb{C}P^N$ such that $\sigma(t) \in G_0$ is represented by

$$\mathbf{diag}(t^{\alpha_0}, \cdots, t^{\alpha_N}), \quad t \in \mathbb{C}^*,$$

where $\alpha_0 \leq \cdots \leq \alpha_N$ are integers. Define a height $\mathbf{h}(M, G_0)$ or simply $\mathbf{h}(G_0)$ to be the smallest $\alpha_N - \alpha_i$ such that $z_i|_M, \cdots, z_N|_M$ have no common zeroes. It is easy to show that M_0 is biholomorphic to M if M_0 is a normal variety and $\mathbf{h}(G_0) = 0$.

The following is proved in [Ti97]

Theorem 4.3. *Let M be a Fano manifold without non-trivial holomorphic vector fields and which admits a Kähler-Einstein metric. Then M is K-stable in the above sense with respect to any very ample $K_M^{-\ell}$.*

This follows from the properness of the K-energy and the computation of its derivative along any given one-parameter subgroup G_0.

For a general polarized Kähler manifold (M, L), we can not expect a uniform bound on diameter for Kähler metrics with constant scalar curvature and with Kähler class $c_1(L)$, consequently, we can not restrict the stability criterion to only those G_0 with normal limit variety M_0. However, we expect that any sequence of Kähler metrics with constant scalar curvature and with bounded Kähler classes contains a subsequence converging to a finite union of complete Kähler manifolds with constant scalar curvature and with finite volume. because of this, it is reasonable to assume that M_0 has no multiple components.

Definition 4.4. *Let (M, L) be a polarized Kähler manifold and $M \subset \mathbb{C}P^N$ by sections of L^ℓ. We say that M is K-semistable with respect to L^ℓ if $\mathbf{w}(G_0) \geq 0$ for $G_0 \subset \mathbf{SL}(N+1)$ such that the corresponding M_0 has no multiple components. We say that M is K-stable with respect to $L^{-\ell}$ if it is K-semistable and $\mathbf{w}(G_0) > 0$ for $G_0 \subset \mathbf{SL}(N+1)$ such that the corresponding M_0 has no multiple components and is not biholomorphic to M.*

We say that (M, L) is asymptotically K-stable if M is K-stable with respect to L^ℓ for sufficiently large ℓ.

Now we recall Donaldson's version of the K-stability [Do02]. It is more algebraic and neater. We say $(\mathcal{M}, \mathcal{L}) \mapsto \mathbb{C}$ a test configuration of a polarized manifold (M, L) if it consists of a scheme $\mathcal{M}$ endowed with a $\mathbb{C}^*$-action that

[16] We assume that M_0 is a normal variety because of the compactness we expect for the set of Kähler metrics with Ricci curvature bounded from below by a positive constant. For those Kähler metrics, the diameter is uniformly bounded and essential singularity should occur only along a closed subset of Hausdorff codimension at least 4.

linearizes on a line bundle $\mathcal{L}$ over $\mathcal{M}$, and a flat $\mathbb{C}^*$-equivariant map $f : \mathcal{M} \mapsto \mathbb{C}$ (where $\mathbb{C}$ has the usual weight one $\mathbb{C}^*$- action) such that $\mathcal{L}|_{f^{-1}(0)}$ is ample on $f^{-1}(0)$ and we have $(f^{-1}(1), \mathcal{L}|_{f^{-1}(1)}) \cong (M; L^r)$ for some $r > 0$. When (M, L) has a $\mathbb{C}^*$-action $\rho : \mathbb{C}^* \mapsto \mathrm{Aut}(M)$, a test configuration where $\mathcal{M} = M \times \mathbb{C}$ and $\mathbb{C}^*$ acts on $\mathcal{M}$ diagonally through ρ is called product configuration. This terminology is given in [Do02]. Also an analogous version of such a test configuration is given in [Ti97] under the name: *A special degeneration.*

Let (V, L) be a n-dimensional polarized variety or scheme and $\mathrm{Aut}(M, L)$ be the group of all automorphisms which can be lifted to L. Given a one parameter subgroup $\rho : \mathbb{C}^* \mapsto \mathrm{Aut}(V)$, we denote by $w(V, L)$ the weight of the $\mathbb{C}^*$-action induced on $\Lambda^{\mathrm{top}} H^0(V, L)$ and then we have the following asymptotic expansions as $\ell \gg 0$:

$$h^0(V, L^\ell) = a_0 \ell^n + a_1 \ell^{n-1} + O(\ell^{n-2}) \tag{4.7}$$

$$w(V, L^\ell) = b_0 \ell^{n+1} + b_1 \ell^n + O(\ell^{n-1}). \tag{4.8}$$

The Donaldson's version of the Futaki invariant of the action is defined as

$$F(V, L; \rho) = \frac{b_1 a_0 - b_0 a_1}{a_0^2}. \tag{4.9}$$

It is shown in [Do02] that $-F(V, L, \rho)$ is equal to the original Futaki invariant (possibly modulo multiplication by a universal positive constant) whenever V is smooth. It is also true for any reduced variety V as shown in [PT04] by using the so called CM line bundles.

Now we can state Donaldson's version of the K-stability.

Definition 4.5. *A polarized Kähler manifold (M, L) is K-semistable if for each test configuration $f : (\mathcal{M}, \mathcal{L}) \mapsto \mathbb{C}$ for (M, L) the Futaki invariant of the induced action on the central fiber $(f^{-1}(0), \mathcal{L}|_{f^{-1}(0)})$ is non-positive. We say (M, L) is K-stable if the Futaki invariant for any test configuration is strictly negative unless we have a product configuration.*

In view of the identification between the original and Donaldson's definition of the Futaki invariant for reduced varieties in [PT04], it is expected that the above two versions of the K-stability are equivalent. Indeed, it follows from a recent work by Arezzo-La Nave-Della Vedova [ALV09] that the two versions of the K-semistability are equivalent.[17]

The importance of the K-stability is partly shown in the following result of J. Stoppa [Sto07].

Theorem 4.6. *Let (M, L) be a polarized Kähler manifold with trivial* $\mathrm{aut}(M, L)$. *Then M admits a Kähler metric with constant scalar curvature with Kähler class $c_1(L)$ only if (M, L) is K-stable.*

[17] Note that signs in two definitions of the K-stability are opposite. This is because we used different orientations in the $\mathbb{C}^*$-actions. Also Arezzo-La Nave-Della Vedova has found a proof in the case of the K-stability.

Its proof is based on Arezzo-Pacard's work [ArPa05] and the works on the K-semi-stability ([ChTi04], [Do05]). In [ChTi04] and [Do05], the authors prove independently that M admits a Kähler metric with constant scalar curvature with Kähler class $c_1(L)$ only if (M, L) is K-semistable. Recently, Stoppa's result has been extended by T. Mabuchi to the general case.

The following well-known conjecture also shows the importance of the K-stability.

Conjecture 4.7. *Let (M, L) be a polarized Kähler manifold. For simplicity, assume that* $\mathrm{Aut}(M, L)$ *is discrete. Then M admits a Kähler metric with constant scalar curvature and Kähler class $c_1(L)$ if and only if (M, L) is asymptotically K-stable.*[18]

4.2 Partial C^0-estimates

In this subsection, I present an approach to solving (1.1) under suitable geometric condition which I have been pursuing since the late 80's. My solution for complex surfaces in the last section can be regarded as a successful example of this approach. Now the geometric condition is much better understood and should be the K-stability.

First we recall that in order to establish the existence of Kähler-Einstein metrics on a Fano manifold M, we only need to establish the a priori C^0-estimate for the solutions of (1.2) for $t \geq t_0$ for some $t_0 > 0$ which may depend on (M, ω):

$$(\omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi)^n \;=\; e^{h_\omega - t\varphi}\omega^n. \tag{4.10}$$

As said before, there are two steps in this approach. First we need to prove a partial C^0-estimate which we have discussed for Del-Pezzo surfaces in a previous section.

Consider the set

$$\mathcal{K}(M, t_0) \;=\; \{\, \tilde{\omega} \,|\, [\tilde{\omega}] = c_1(M),\ \mathrm{Ric}(\tilde{\omega}) \;\geq\; t_0\tilde{\omega} \,\}.$$

For any $\tilde{\omega}$, choose a Hermitian metric $\tilde{h}$ with $\tilde{\omega}$ as its curvature form and any orthonormal basis $\{S_i\}_{0\leq i\leq N}$ of each $H^0(M, K_M^{-\ell})$ with respect to an induced inner product. Put

$$\rho_{\tilde{\omega},\ell}(x) \;=\; \sum_{i=0}^{N} ||S_i||^2_{\tilde{h}}(x). \tag{4.11}$$

This is independent of the choice of $\tilde{h}$ and the orthonormal basis $\{S_i\}$.

[18] This conjecture is often referred to the Yau-Tian-Donaldson conjecture. In general, one can expect that (M, L) admits a constant scalar curvature metric if and only if M is asymptotically weakly K-stable. The weak K-stability means that M is semistable and $\mathbf{w}(G_0) > 0$ for any G_0 from Definition 4.4 which is transverse to the identity component of $\mathrm{Aut}(M, L)$ in a suitable sense. We refer the readers to recent works of Mabuchi for details.

Conjecture 4.8. *[Ti90] There are uniform constants $c_k = c(k,n) > 0$ for $k \geq 1$ and $\ell_i \to \infty$ with $i \geq 0$ and $\ell_0 = \ell_0(n)$ such that for any $\tilde\omega \in \mathcal{K}(M,t_0)$ and $\ell = \ell_i$ for each i,*

$$\rho_{\tilde\omega,\ell} \geq c_\ell > 0. \tag{4.12}$$

Remark 4.9. *There is a uniform upper bound on $\rho_{\tilde\omega,\ell}$ which depends only on n and ℓ. This is proved in [Ti90]. The lower bound is more important and can be regarded as an effective version of the very ampleness in algebraic geometry.*

Remark 4.10. *A stronger version of Conjecture 4.8 may hold: There are uniform constants $c_k = c(k,n) > 0$ for $k \geq 1$ and $\ell_0 = \ell_0(n)$ such that for any $\tilde\omega \in \mathcal{K}(M,t_0)$, and $\ell \geq \ell_0$, $\rho_{\tilde\omega,\ell} \geq c_\ell$.*

It is known that if φ is a solution of (1.2) ($t \geq t_0$), then $\omega_t := \omega_\varphi$ has its Ricci curvature greater than or equal to $t \geq t_0$. If Conjecture 4.8 is true, Then as we did for (3.6), we can deduce

$$||\varphi - \sup_M \varphi - \frac{1}{\ell}\log(\sum_{i=0}^N \lambda_i^2 ||\sigma_i||^2)||_{C^0} \leq C, \tag{4.13}$$

where $||\cdot||$ is a Hermitian metric whose curvature is ω and $\{\sigma_i\}$ is an orthonormal basis of $H^0(M, K_M^{-\ell})$ with respect to the inner product induced by $||\cdot||$.

For each orthonormal basis $\{S_i\}$ of $H^0(M, K_M^{-\ell})$, we have embedding $\Phi : M \mapsto \mathbb{C}P^N$. Such an embedding is unique modulo the isometry group $U(N+1)$ of $\mathbb{C}P^N$. If ω_i is any sequence in $\mathcal{K}(M,t_0)$ and Φ_i is corresponding embedding, then by taking a subsequence of necessary, there is a limit cycle $\bar M_\infty \subset \mathbb{C}P^N$ of $\Phi_i(M)$. The partial C^0-estimate (4.12) is closely related to the irreducibility of this limit cycle $\bar M_\infty$. Under suitable regularity of the metric limits of (M, ω_i), they are equivalent.

On the other hand, since we have $\mathrm{Ric}(\omega_i) \geq t_0\omega_i$ and $[\omega_i] = c_1(M_i)$, it contains a subsequence converging to a length space (M_∞, d_∞) in the Gromov-Hausdorff topology. Note that the diameter of (M_∞, d_∞) is uniformly bounded. I expected the following [19]

Conjecture 4.11. *The above Gromov-Hausdorff limit M_∞ can be identified with the complex limit $\bar M_\infty$.*

We can say more about it in the case that ω_i is a sequence of Kähler-Einstein metrics with $\mathrm{Ric}(\omega_i) = \omega_i$ since we can apply certain works of Cheeger-Colding-Tian [CCT95]. We may assume that there is a Gromov-Hausdorff limit (M_∞, d_∞) of Kähler-Einstein manifolds (M_i, ω_i) . The main theorem in [CCT95] gives partial regularity for this limit.

[19]It had come to my attention more than sixteen years ago. I have made many attempts to solving this for a sequence of Kähler-Einstein metrics.

Theorem 4.12. *[CCT95] Let (M_i, ω_i) be a sequence of Kähler-Einstein manifolds with $\mathrm{Ric}(\omega_i) = \omega_i$ and which converges to (M_∞, d_∞) in the Gromov-Hausdorff topology. Then there is a closed subset $S \subset M_\infty$ of Hausdorff codimension at least 4 such that $M_\infty \backslash S$ is a smooth Kähler manifold and d_∞ is induced by a Kähler-Einstein metric ω_∞ outside S with $\mathrm{Ric}(\omega_\infty) = \omega_\infty$. Moreover, ω_i converges to ω_∞ in the C^∞-topology outside S.*

In fact, even though $(M_\infty, \omega_\infty)$ may have singularity, one can still do integration as well as other analysis on M_∞. It was conjectured that the M_∞ *is a variety and S is a subvariety* (cf. [CCT95]). Indeed, J. Cheeger proved that S is rectifiable. The partial C^0-estimate follows from our understanding of the singular set.

Even though S is not completely understood, we can treat M_∞ as a "good" variety in many ways. By an element of $H^0(M_\infty, K_{M_\infty}^{-\ell})$, we mean a holomorphic section S of $K_{M_\infty}^{-\ell}$ on $M_\infty \backslash S$ with finite L^2-norm [20] Then one can prove that the space $H^0(M_\infty, K_{M_\infty}^{-\ell})$ is of finite dimension. Choose a Hermitian metric h_∞ of $K_{M_\infty}^{-1}$ outside S with ω_∞ as its curvature. We may also choose h_i for $K_{M_i}^{-1}$ such that h_i converges to h_∞ outside S. The 2-dimensional version of the following proposition was proved in [Ti89] and the same proof works for higher dimensions.

Proposition 4.13. *By taking a subsequence if necessary, for each ℓ, we have that $H^0(M_i, K_{M_i}^{-\ell})$ converges to $H^0(M_\infty, K_{M_\infty}^{-\ell})$ as i tends to ∞ in the sense: There are orthonormal bases $\{S_a^i\}_{0 \le a \le N}$ of $H^0(M_i, K_{M_i}^{-\ell})$ with respect to h_i such that S_a^i converges to S_a^∞ $(0 \le a \le N)$ as i tends to ∞ and $\{S_a^\infty\}$ forms an orthonormal basis of $H^0(M_\infty, K_{M_\infty}^{-\ell})$.*

Proof. We outline its proof here. As in [Ti89], the proof uses the L^2-estimate for $\bar\partial$-operator and the theory for elliptic equations. First we observe: For each i,

$$\Delta_{\omega_i} ||S_a^i||^2 = ||\nabla S_a^i||^2 - \ell\, ||S_a^i||^2.$$

Then we can apply the Moser iteration to deriving a uniform bound on $||S_a^1||$ as well as bounds on derivatives of S_a^i outside the singular set of M_∞. It follows that by taking a subsequence if necessary, we can assume that S_a^i converges to a S_a^∞ as i tends to ∞.

To complete the proof, we need to show that each section $\tilde S_\infty$ of $K_{M_\infty}^{-\ell}$ on M_∞ is the limit of a sequence of sections $\tilde S_i$ on M_i. This is done by using Hömander's L^2-estimate. For any $\epsilon > 0$, choose a finite cover $\{B_{r_k}(x_k)\}$ of the singular set $S \subset M_\infty$ satisfying:

1. $r_k \le \epsilon$;
2. $\sum_k r_k^{2n-4} \le C < \infty$;[21]
3. For any $x \in M_\infty$, the number of balls $B_{r_k}(x_k)$ containing x is uniformly bounded, say C.

[20] This should be automatically true since S is of codimension at least 4.

[21] We will always use C to denote a uniform constant in this proof, though its actual value may vary in different places.

Let η be a smooth function on $\mathbb{R}$ such that $\eta(r) = 0$ for $r \leq 1$ and $\eta(r) = 1$ for $r \geq 2$. Put

$$S'_\infty = \left(\prod_k \eta(d(\cdot, x_k)/r_k)\right) \tilde{S}_\infty.$$

Then one can show

$$\int_{M_\infty} ||\bar{\partial} S'_\infty||^2 \, \omega_\infty^n \leq C\epsilon^2.$$

For i sufficiently large, there is a diffeomorphism $\Phi_i : M_i \backslash K_i \mapsto M_\infty \backslash S$ satisfying:
(1) The measure of K_i is bounded by $C\epsilon^4$;
(2) The image of Φ_i contains the complement of $\cup_k B_{r_k}(x_k)$ in M_∞;
(3) $||\bar{\partial}\Phi_i|| \leq \epsilon$.

Each Φ_i induces an isomorphism τ_i between the complex line bundles $K_{M_i}^{-\ell}$ and $K_{M_\infty}^{-\ell}$ such that $||\bar{\partial}\tau_i|| \leq C_\ell \epsilon$, where C_ℓ is independent of i and ϵ. Put $S'_i = \tau^{-1}(S'_\infty)$. Then

$$\int_{M_i} ||\bar{\partial} S'_i||^2 \, \omega_i^n \leq C_\ell \, \epsilon^2.$$

By applying Hömander's L^2-estimate for $\bar{\partial}$-operators, we can have a C^∞-section u_i of $K_{M_i}^{-\ell}$ satisfying:

$$\bar{\partial} u_i = -\bar{\partial} S'_i, \qquad \int_{M_i} ||u_i||^2 \, \omega_i^n \leq C_\ell \, \epsilon^2.$$

It implies that $\tilde{S}_i = S'_i + u_i$ is a holomorphic section of $K_{M_i}^{-\ell}$ such that $\tilde{S}_i$ is within the $C\epsilon$-distance of $\tilde{S}_\infty$ for some uniform constant C. Then the proposition follows easily. □

Proposition 4.13 can be regarded as a metric version of the flatness for families of projective varieties.

Now we fix a large ℓ and consider embedding $\Phi_i(M) \subset \mathbb{C}P^N$ of M_i induced by $\{S_a^i\}$.

Proposition 4.14. *Assume that $\Phi_i(M_i)$ converges to an irreducible subvariety $\bar{M}_\infty$. Then there is a uniform constant c satisfying: $\rho_{\omega_i,\ell} \geq c$ for all i, i.e.,* (4.12) *holds for all i.*

Proof. We just outline the proof. First we observe

$$0 \leq \omega_i = \tilde{\omega}_i - \frac{\sqrt{-1}}{2\ell} \partial\bar{\partial} \log(\rho_{\omega_i,\ell}),$$

where $\tilde{\omega}_i$ is the restriction of ω_{FS} to $\Phi_i(M)$. In particular,

$$\Delta_{\tilde{\omega}_i} \left(-\log(\rho_{\omega_i,\ell})\right) \geq -n.$$

Since $\bar{M}_\infty$ is irreducible, we have uniform bounds on the Sobolev constant and the Poincare constant for $(\Phi_i(M), \tilde{\omega}_i)$. Then the standard Moser iteration implies

$$-\inf_{M_i} \log(\rho_{\omega_i,\ell}) \le C\left(1 + \int_{M_i} |\log(\rho_{\omega_i,\ell})| \tilde{\omega}_i^n\right).$$

On the other hand, by the above proposition, we know that $\rho_{\omega_i,\ell}$ converges to $\rho_{\omega_\infty,\ell}$ in the C^∞-topology outside S, so they are uniformly bounded on any compact subsets away from S. Then one can easily deduce from the above $\rho_{\omega_i,\ell} \ge c$ for some uniform positive constant. □

Remark 4.15. *The converse is true: If* (4.12) *holds, then* $\bar{M}_\infty$ *is irreducible.*

Corollary 4.16. *Let* M_i *and* $\bar{M}_\infty$ *be as above. If* $\bar{M}_\infty$ *is irreducible, then* $\bar{M}_\infty$ *is the Gromov-Hausdorff limit of* (M_i, ω_i)*. In particular, Conjecture 4.11 holds.*

It shows how important to have the irreducibility of the complex limits of $\Phi_i(M_i)$ under embedding given by $H^0(M_i, K_{M_i}^{-\ell})$. Also I would like to point out that the irreducibility can be derived if limits of the embeddings Φ_i stabilize for sufficiently large ℓ. This gives a strong evidence on the irreducibility. Let me elaborate a bit more: For all sufficiently large ℓ, we have embedding $\Phi_{i,\ell} : M_i \mapsto \mathbb{C}P^{N_\ell}$ [22] and get a limit $\bar{M}_{\infty,\ell}$ of $\Phi_{i,\ell}(M_i)$ as i tends to ∞ (possibly after taking a subsequence). On the other hand, $\Phi_{i,\ell}$ converges to a limit map (possibly rational and after taking a subsequence):

$$\Phi_{\infty,\ell} : M_\infty \backslash S \mapsto \mathbb{C}P^{N_\ell}.$$

For any tubular neighborhood U of S, one can prove that for ℓ sufficiently large, $\Phi_{\infty,\ell}$ is an embedding on $M_\infty \backslash U$. Hence, for ℓ and ℓ' large, $\bar{M}_{\infty,\ell}$ can be identified with $\bar{M}_{\infty,\ell'}$ by a biholomorphic map outside a small closed subset. This means that $\Phi_{\infty,\ell}$ stabilize modulo a small subset. It may be sufficient for deducing the irreducibility and consequently, the partial C^0-estimate. I will give more analysis on this in a separate paper.

Remark 4.17. *In view of my joint work with J. Viaclovsky [TV05], one can expect that Theorem 4.12 also holds for Kähler metrics with constant scalar curvature and bounded diameter. If so, following the same arguments as we did above, one can also derive a partial* C^0*-estimate for potential functions for Kähler metrics with constant scalar curvature and bounded diameter. This may be used in proving the convergence of K-stable Kähler manifolds with constant scalar curvature.*

4.3 Relating K-stability to existence

In this subsection, we sketch how to deduce the existence of Kähler-Einstein metrics from the K-stability under the assumption of the partial C^0-estimate

[22] This $\Phi_{i,\ell}$ is the Φ_i used above. Now we insert an extra subscript to emphasis dependence on ℓ.

(4.8). These arguments were discussed before in many of my lectures. It may be useful to say a few words on them. More details can be provided later. I would like to point out that the arguments in this subsection work for general Kähler metrics with constant scalar curvature if there is a suitable version of partial C^0-estimate as we discussed at the end of the last subsection.

In order to establish the existence, we need a priori C^0-estimates on solutions for (4.10). Let φ_i be a solution for (4.10) for $t_i \geq t_0$. Write $\omega_i = \omega_{\varphi_i}$. We assume that the partial C^0-estimate holds for ω_i for a sufficiently large ℓ and M is K-stable with respect to the embedding given by $H^0(M, K_M^{-\ell})$. We want to show that φ_i are uniformly bounded.

First we observe that the K-energy $\mathbf{T}_\omega$ is monotonically decreasing along (4.10), so we have a uniform bound

$$\mathbf{T}_\omega(\varphi_i) \leq c = c(\omega). \tag{4.14}$$

We may assume that $M \subset \mathbb{C}P^N$ through an embedding given by an orthonormal basis of $H^0(M, K_M^{-\ell})$ with respect to ω. By our assumption on the partial C^0-estimate for ω_i, there is a $\sigma_i \in \mathbf{SL}(N+1)$ such that

$$\frac{1}{\ell}\sigma_i^* \omega_{FS} = \omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\psi_i, \quad ||\psi - \varphi_i||_{C^0} \leq C. \tag{4.15}$$

Here C denotes a uniform constant. In fact,

$$\psi - \varphi_i = \frac{1}{\ell}\log\left(\sum_{a=0}^N ||S_a^i||_i^2\right),$$

where $||\cdot||_i$ is a Hermitian norm whose curvature is ω_i and $\{S_a^i\}$ is an orthonormal basis with respect to ω_i. It follows

$$\mathbf{T}_\omega(\psi_i) \leq c' = c'(\omega). \tag{4.16}$$

For simplicity, we assume that M has no non-trivial holomorphic fields. If φ_i are not uniformly bounded, $\sigma_i(M)$ converges to a variety which is not biholomorphic to M. [23] For each i, join $\mathrm{I} \in \mathbf{SL}(N+1)$ to σ_i by the orbit O_i of a C^*-action, without loss of generality, we may assume that O_i converge to a C^*-orbit O_∞. Using appropriate compactification of $\mathbf{SL}(N+1)(M)$, one can show that if $\sigma(e^t)$ $(t \in \mathbb{C})$ is the limit C^*-action, $\sigma(e^t)(M)$ converge to the limit of $\sigma_i(M)$ as t tends to ∞. Then the K-stability assumption for M implies that $\mathbf{T}_\omega(\varphi_{\sigma(t)})$ diverges to ∞ as t tends to ∞. Next one shows that there are $t_i \to \infty$ satisfying:

$$||\psi_i - \varphi_{\sigma(e^{t_i})}||_{C^0} \leq C.$$

Then $\mathbf{T}_\omega(\psi_i)$ are unbounded, a contradiction. Hence, φ_i should stay bounded and there is a Kähler-Einstein metric on M.

[23] If M has non-trivial holomorphic fields, we can modify σ_i by automorphisms of M so that this still holds.

The same arguments also show the following: If $(M_i, \tilde{\omega}_i)$ is a sequence of Kähler manifolds with $c_1(M_i) = [\tilde{\omega}_i]$ converging to a Kähler manifold $(M_\infty, \omega_\infty)$ and M_∞ is asymptotically K-stable, we further assume that M_i admit Kähler-Einstein metrics ω_i with $c_1(M_i) = [\omega_i]$ for which the partial C^0-estimate holds, then ω_i converges to a Kähler-Einstein metric on M_∞. This is exactly what we did for complex surfaces in [Ti89].

4.4 Kähler-Ricci flow on Fano manifolds

In this subsection, we discuss some results on the Kähler-Ricci flow on a Fano manifold M. First we recall that the Kähler-Ricci flow (3.18) on M has a global solution $\varphi(t)$ for $t \geq 0$, i.e., $\varphi(t)$ satisfies:

$$\frac{\partial \varphi}{\partial t} = \log \left(\frac{(\omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi)^n}{\omega_0^n} \right) + h_0 + \varphi, \quad \varphi(0) = 0, \tag{4.17}$$

where $h_0 = h_{\omega_0}$. Moreover, if $||\varphi(t)||_{C^0}$ is uniformly bounded, then $\varphi(t)$ converges to a smooth function φ_∞ such that ω_{φ_∞} is a Kähler-Einsetin metric. However, because of known obstructions, we can not expect a C^0-estimate for general Fano manifolds. What one can expect is to get the C^0-estimate for K-stable Fano manifolds.

Let $\varphi(t)$ be the global solution of (4.17) $(t \geq 0)$ and $\omega_t = \omega_{\varphi(t)}$. A folklore conjecture states: *ω_t converges to a Kähler-Ricci soliton (possibly with "mild" singularities of codimension at least 4) in the Cheeger-Gromov topology.* Partial progresses on this conjecture have been made in recent years. The following theorem is due to G. Perelman (cf. [SeTi07]):

Theorem 4.18. *We have*
(1) The scalar curvature of ω_t is uniformly bounded;
(2) The diameter of ω_t is uniformly bounded;
(3) The time derivative $\dot{\varphi}$ of $\varphi(t)$ is uniformly bounded if we normalize $\varphi(t)$ by

$$\int_M \left(e^{-\varphi(t)} - 1 \right) \omega_t^n = 0.$$

Using this theorem, one can verify the above conjecture if one can further bound the Ricci curvature of ω_t (cf. [SeTi07]). If one can further bound the curvature, one can even prove that the limit soliton has no singularity, though its complex structure may differ from that of M. The above conjecture follows if one can extend the works of Cheeger-Colding on space of metrics with bounded Ricci curvature to the Ricci flow on a compact manifold with bounded diameter, scalar curvature as well as bounded Sobolev constants.

In the opposite direction, it is known that if the Fano manifold M admits a Kähler-Einstein metric, then the solution $\varphi(t)$ for (4.17) converges to a φ such that ω_φ is Kähler-Einstein (cf. [Pe03], [TZ07]). In [TZ07], this convergence result is extended to the case when M admits only a Kähler-Ricci soliton.

5 Appendix A: Finite dimensional α-invariants

In this appendix, we discuss a sequence of holomorphic invariants which were first introduced in my Harvard PhD thesis [Ti88] (also see [Ti89]). The motivation then was for solving a conjecture of Calabi for Del-Pezzo surfaces. There has been much progress on computing such invariants and understanding their relation to $\alpha(M)$ in [Ti87]. It was shown later that they have algebraic correspondences: log canonical thresholds in the study of projective manifolds and resolution of singularity.

Assume that (M, L) is a polarized projective manifold of dimension n. [24] Fix a Hermitian metric h on L such that its curvature ω gives a Kähler metric on M, then they induce a Hermitian inner product on $H^0(M, L)$. We can define a function $\rho_{L,h,k}$ on the Grassmannian manifold $M \times G(k, H^0(M, L))$ as follows: For any k-dimensional subspace $P \in G(k, H^0(M, L))$,

$$\rho_{L,h,k}(x, P) := \sum_{i=1}^{k} ||S_i||_h^2(x),$$

where $\{S_i\}$ is any orthonormal basis of P with respect to the induced inner product. One can easily show that this function is well-defined, i.e., it is independent of the choice of the orthonormal basis. Now we define

$$\alpha_{\ell,k}(M, L) := \sup\{\alpha \mid \sup_P \int_M \left(\frac{1}{\rho_{L^\ell,h^\ell,k}(\cdot, P)}\right)^{\frac{\alpha}{\ell}} \omega^n < \infty \}. \tag{5.1}$$

One can prove that this is an invariant of (M, L), i.e., it is independent of the choice of h. Moreover, we have

$$0 < c \le \alpha_{\ell,1}(M, L) \le \alpha_{\ell,2}(M, L) \le \cdots \le \alpha_{\ell,k}(M, L) \le \cdots,$$

where c depends only on (M, L). When $L = K_M^{-1}$, we simply denote them by $\alpha_{\ell,k}(M)$. In Section 3, we have seen that they are closely related to the existence of Kähler-Einstein metrics.

There is an algebraic counterpart of $\alpha_{\ell,k}(M, L)$. For simplicity, we assume $k = 1$. Recall that we say a pair (M, D), where D is an effective divisor, is log canonical if there is a blow-up $\pi : M' \mapsto M$ such that $D' := \sum_i E_i + \pi_*^{-1} D$, where $\pi_*^{-1} D$ is the quadratic transformation of D, is a normal crossing and $K_{M'} + D' = \pi^*(K_M + D) + \sum_i a_i E_i$, where $a_i \ge -1$ and E_i are exceptional divisors of M'. The log canonical thresfold of any pair (M, D) is defined by

$$lct(M, D) = \sup\{ \lambda \in \mathbb{Q} \mid (M, \lambda D) \text{ is log canonical } \}. \tag{5.2}$$

The global log canonical threshold $lct(M, L)$ of (M, L) is defined as

$$lct(M, L) = \inf\{ lct(M, D) \mid D \in |L| \}. \tag{5.3}$$

[24] In [Ti88], M is assumed to be Fano and L is simply the anti-canonical bundle. It is not hard to see that this restriction is unnecessary (also see [Ch08]).

The log canonical threshold was introduced by V. Shokurov [Sho92] and has many applications in the classification of projective manifolds. It can be identified with $\alpha_{\ell,1}(M,L)$, more precisely, one can prove (cf. [Ch07])

$$\alpha_{\ell,1}(M,L) \;=\; \frac{1}{\ell}\, lct(M,L^{\ell}), \quad \forall \ell \geq 1. \tag{5.4}$$

In particular, if M is Fano, we have

$$\alpha_{\ell,1}(M) \;:=\; \alpha_{\ell,1}(M,K_M^{-1}) \;=\; lct(M,K_M^{-\ell}))/\ell.$$

In [Ch07] and [Ch08], I. Cheltsov computes $lct(M,L)$ for Del-Pezzo surfaces and many Fano manifolds of higher dimensions. In particular, if M is a smooth Del-Pezzo surface with $K_M^2 \leq 4$, he shows

$$lct(M,K_M^{-\ell}) \;=\; \ell\, lct(M,K_M^{-1}).$$

He also estimates them from below. Identifying log canonical thresholds with $\alpha_{\ell,1}(M)$, we get

Theorem 5.1. *(I. Cheltsov) Let M be a smooth Del-Pezzo surface with $K_M^2 \leq 3$ and without any Eckardt points, then*

$$\alpha_{\ell,1}(M) \;=\; \alpha_{1,1}(M) \;\geq\; \frac{3}{4}.$$

There are fewer results on $\alpha_{\ell,k}(M)$. In [Ti89], I proved that $\alpha_{6,2}(M) > 2/3$ for any smooth cubic surface $M \subset \mathbb{C}P^3$. In [Shi09], Y.L. Shi proved the following:

Theorem 5.2. *(Y.L. Shi) Let M be a smooth cubic surface in $\mathbb{C}P^3$. Then $\alpha_{\ell,2}(M) > 2/3$.*

This allows him to give an elegant and simpler proof of my theorem on the existence of Kähler-Einstein metrics on Del-Pezzo surfaces in [Ti89] (see [Shi09] and also section 3).

In general, it is a difficult task to compute $\alpha_{\ell,k}(M,L)$ and understand relations among $\alpha_{\ell,k}$ for a fixed k. The following conjecture was proposed by myself a while ago:

Conjecture 5.3. *For any polarized manifold (M,L) and $k \geq 1$, there is an $\ell_0 > 0$ such that for all $\ell \geq \ell_0$,*

$$\alpha_{\ell,k}(M,L) \;=\; \alpha_{\ell_0,k}(M,L).$$

We can further speculate

Conjecture 5.4. *We can ask the stronger version of Conjecture 5.3: If the ring $R(M,L) := \oplus_{\ell\geq 0} H^0(M,L^{\ell})$ is generated by $H^0(M,L)$, then for $k \in [1,k_0]$,*

$$\alpha_{\ell,k}(M,L) = \alpha_{1,k}(M,L),$$

where $k_0 \geq 1$ depends only on M and L. More generally, if $R(M,L)$ is generated by $\oplus_{\ell=0}^{\ell_0} H^0(M,L^{\ell})$, then for $\ell \geq \ell_0$ and small k,

$$\alpha_{\ell,k}(M,L) = \alpha_{\ell_0,k}(M,L).$$

If G is a compact subgroup of the automorphism group $\mathrm{Aut}(M,L)$ of (M,L), then we can easily extend the above construction to have G-invariant versions $\alpha_{\ell,k,G}(M,L)$ of $\alpha_{\ell,k}(M,L)$. For this purpose, we only need to assume that h and P in (5.1) are G-invariant.

In applications of α-invariants, we often need a uniform bound on the integral in (5.1). More precisely, we have

Theorem 5.5. *[PhSt00], [DeKo01] Assume that (M_j, L_j) be a sequence of polarized manifolds converging to a polarized manifold (M_∞, L_∞) and $P_j \in G(k, H^0(M_j, L_j^\ell))$ converge to $P_\infty \in G(k, H^0(M_\infty, L_\infty))$. Let h_j be Hermitian metrics on L_j converging to a Hermitian metric h_∞ on L_∞ such that curvature ω_∞ of h_∞ is a Kähler metric. Then for any $\alpha < \alpha_{\ell,k}(M_\infty, L_\infty)$, there is a uniform constant C such that*

$$\int_{M_j} \left(\frac{1}{\rho_{L_j^\ell, h_j^\ell, k}(\cdot, P_j)} \right)^\alpha \omega_j^n \,\le\, C, \tag{5.5}$$

where ω_j is the curvature form of h_j.

The special case of this theorem for $n = 2$ was first given and shown in the appendix of [Ti89]. The motivation is to derive the properness of $\mathbf{F}_{\omega_\tau}$ on the subset of functions in $P(M_\tau, \omega_\tau)$ induced by sections in $H^0(M_\tau, K_{M_\tau}^{-6})$ as shown in Section 3. Note that the proof of Theorem 5.5 is essentially local and can be reduced to a similar problem on a sequence of holomorphic functions.

Finally, we recall the definition of $\alpha(M,L)$: Choose any Kähler metric ω with Kähler class $2\pi c_1(L)$. Then

$$\alpha(M,L) \,=\, \inf\{\alpha \mid \sup_{\varphi \in P(M,\omega)} \int_M e^{-\alpha(\varphi - \sup_M \varphi)}\, \omega^n \,<\, \infty\,\} \tag{5.6}$$

If $G \subset \mathrm{Aut}(M,L)$ is a compact subgroup, then

$$\alpha_G(M,L) \,=\, \inf\{\alpha \mid \sup_{\varphi \in P_G(M,\omega)} \int_M e^{-\alpha(\varphi - \sup_M \varphi)}\, \omega^n \,<\, \infty\,\} \tag{5.7}$$

Clearly, $\alpha_{\ell,1}(M,L) \ge \alpha(M,L)$ and $\alpha_{\ell,1,G}(M,L) \ge \alpha_G(M,L)$. The following theorem gives a closer relation among them.

Theorem 5.6. *For any polarized manifold (M,L), we have*

$$\alpha(M,L) \,=\, \lim_{\ell \to \infty} \alpha_{\ell,1}(M,L), \quad \alpha_G(M,L) \,=\, \lim_{\ell\to\infty} \alpha_{\ell,1,G}(M,L).$$

This was shown by Demailly (see [Ch08], Appendix) and by Y.D. Shi, independently, in his PhD thesis (see [Shi09]).

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