SIMON MARAIS mathematics competition

2022 PAPER A: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions. Tablets may be used solely for writing worked solutions, with internet access switched off.
- For participants sitting the exam off-site, an electronic device such as a PC, laptop, phone or tablet may be used during the competition for accessing the papers, undergoing invigilation, writing and submitting solutions and (for pairs entrants) communicating with the other member of the pair.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "A3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and the paper (as in **S1234567A** (for singles) or **P3141593A** (for pairs)), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

SIMON MARAIS mathematics competition

2022 PAPER A: PROBLEMS

A1. Let $ABCD$ be a unit square, and let P be a point inside triangle ABC. Let E and F be the points on AB and AD, respectively, such that $AEPF$ is a rectangle. Let x and y denote the lengths of AE and AF, respectively.

Determine a two-variable polynomial $f(x, y)$ with the property that the area of $AEPF$ is greater than the area of each of the quadrilaterals $EBCP$ and $FPCD$ if and only if $f(x, y) > 0$.

A2. Let n be a positive integer, and let S be a set with 2^n elements. Let $A_1, A_2, \ldots, A_n$ be randomly and independently chosen subsets of S, where each possible subset of S is chosen with equal probability. Let P_n be the probability that

$$A_1 \cup A_2 \cup \cdots \cup A_n = S \qquad \text{and} \qquad A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset.$$

Prove that $\lim_{n\to\infty} P_n = \dfrac{1}{e^2}$.

A3. Let $0 < a < 1$ be a fixed real number. Show that there are at least two values of x in the interval $(0, 1)$ such that

$$\int_0^x \left(\sin\left(\frac{\pi \sin \frac{\pi t}{2}}{2} \right) + \frac{2}{\pi} \arcsin\left(\frac{2}{\pi} \arcsin t \right) - 2t \right) \mathrm{d}t = \frac{1}{2}\left(a - \frac{2}{\pi} \arcsin\left(\frac{2}{\pi} \arcsin a \right) \right) \left(\sin\left(\frac{\pi \sin \frac{\pi a}{2}}{2} \right) - a \right).$$

A4. Let n be a positive integer, and let $q \geq 3$ be an odd integer such that every prime factor of q is larger than n. Prove that

$$\frac{1}{n!\,(q-1)^n} \prod_{i=1}^{n} (q^i - 1)$$

is an integer that has no prime factor in common with $\dfrac{q-1}{2}$.

