SIMON MARAIS mathematics competition

2022 PAPER B: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem B1–B3 is worth 7 points. Points for problem B4 are as indicated.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions. Tablets may be used solely for writing worked solutions, with internet access switched off.
- For participants sitting the exam off-site, an electronic device such as a PC, laptop, phone or tablet may be used during the competition for accessing the papers, undergoing invigilation, writing and submitting solutions and (for pairs entrants) communicating with the other member of the pair.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "B3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and the paper (as in **S1234567B** (for singles) or **P3141593B** (for pairs)), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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2022 PAPER B: PROBLEMS

B1. Let $A > 1$ be a real number. Determine all pairs (m, n) of positive integers for which there exists a positive real number x such that $(1 + x)^m = (1 + Ax)^n$.

B2. Let a, b, and c be real numbers, and let P be the polynomial

$$P(x) = x^6 + ax^4 + bx^2 + c.$$

Suppose that there is a unique circle Γ in the complex plane such that all of the roots of P lie on Γ. Prove that $b^3 = a^3c$.

B3. Ari and Sam are playing a game in which they take turns breaking a block of chocolate in two and eating one of the pieces. At each stage of the game the block of chocolate is a rectangle with integer side lengths. On each player's turn, they break the block of chocolate into two such rectangles along a horizontal or vertical line, and eat the piece with smaller area. (If the two pieces have the same area they may eat either one.) The game ends when the block of chocolate is a 1×1 rectangle, and the winner is the last player to take their turn breaking the chocolate in two.

At the start of the game the block of chocolate is a 58×2022 rectangle. If Ari goes first, which player has a winning strategy?

B4. *The following problem is open in the sense that the answer to part (b) is not currently known. A solution to part (a) will be awarded 7 points. Up to 7 additional points may be awarded for progress on part (b).*

Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be two sequences of positive integers, satisfying $a_0 b_0 \geq 600$ and

$$a_{n+1} = a_n + 2 \cdot \lfloor b_n/20 \rfloor,$$
$$b_{n+1} = b_n + 3 \cdot \lfloor a_n/30 \rfloor,$$

for all $n \geq 0$.

(a) Prove that there exists a nonnegative integer N such that

$$-13 \leq a_n - b_n \leq 23,$$

for all $n \geq N$.

(b) Must there exist a nonnegative integer n such that $a_n = b_n$?

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

